Graded contractions of the Pauli graded $sl(3,\mathbb{C})$

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Abstract

The Lie algebra $sl(3,\mathbb{C})$ is considered in the basis of generalized Pauli matrices. Corresponding grading is the Pauli grading here. It is one of the four gradings of the algebra which cannot be further refined.

The set $\mathcal{S}$ of 48 contraction equations for 24 contraction parameters is solved. Our main tools are the symmetry group of the Pauli grading of $sl(3,\mathbb{C})$, which is essentially the finite group $SL(2,\mathbb{Z}_3)$, and the induced symmetry of the system $\mathcal{S}$. A list of all equivalence classes of solutions of the contraction equations is provided. Among the solutions, 175 equivalence classes are non-parametric and 13 solutions depend on one or two continuous parameters, providing a continuum of equivalence classes and subsequently continuum of non-isomorphic Lie algebras. Solutions of the contraction equations of Pauli graded $sl(3,\mathbb{C})$ are identified here as specific solvable Lie algebras of dimensions up to 8. Earlier algorithms for identification of Lie algebras, given by their structure constants, had to be made more efficient in order to distinguish non-isomorphic Lie algebras encountered here.

Resulting Lie algebras are summarized in tabular form. There are 88 indecomposable solvable Lie algebras of dimension 8, 77 of them being nilpotent. There are 11 infinite sets of parametric Lie algebra which still deserve further study.

Key words: Lie algebra, $sl(3,\mathbb{C})$, Pauli grading, graded contraction, Lie algebra identification
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1 Introduction

Study of homomorphic relations between pairs of Lie algebras, called algebra–subalgebra pairs, over complex or real number field, was recognized as a nat-

urally interesting and useful problem with the advent of Lie theory over a century ago. During that century the number and diversity of applications of Lie theory in mathematics, science, and engineering was rapidly increasing. In parallel, it was becoming increasingly more important to know all maximal subalgebras of Lie algebras of practical interest. Lie algebras of semisimple type were classified a century ago in the work of Killing and Cartan. Two important families of their maximal subalgebras were classified during 1950's by Borel and de Siebental [1] (maximal rank reductive subalgebras) and by Dynkin (maximal semisimple subalgebras) [2].

Quite different is the situation with solvable Lie algebras. Classification of the bewildering variety of isomorphism classes of such algebras remains today completely out of reach; only the lowest dimensional ones (≤ 5) have been described so far. A curious sideline of our results illustrates the difficulty: In the second part of this work, we describe over 80 non–isomorphic solvable indecomposable Lie algebras of dimension 8. Compare that with a single isomorphism class of semisimple Lie algebras of that dimension!

Introduction of non-homomorphic relations between Lie algebras was motivated by the practical need to relate meaningfully other pairs of Lie algebras. We are able to point out only two very different relations of such kind, only the second one of the two is of interest to us here. First is the subjoining of Lie algebras (see [3,4,5] and references therein). The second one is deformation of Lie algebras, introduced independently in mathematics [6] and in physics [7]. Intuitively deformations can be defined as non-equivalent transformations of structure constants. For rigid Lie algebras, like all the semisimple ones, that means singular transformations of the constants. Subsequent development of the deformation theory in mathematics followed a different path [8] than in physics, showing so far only a marginal influence on the latter.

In the physics literature there are several hundreds of papers dealing with deformation/contraction of Lie algebras. Although their critical review would be a very timely project, it is far beyond the scope of this article. Our immediate goal is to apply recently invented variant of Wigner-Inönü contractions, called graded contractions [9], to a specific case, the Lie algebra $sl(3,\mathbb{C})$ equipped with the Pauli grading [10], and to find the outcome of all contractions preserving that grading.

The set of $n \times n$ matrices, called here 'generalized Pauli matrices' [10], has been known in mathematics long before W. Pauli as a curious associative algebra, or as a finite group of order n^3 [11,12,13]. In the physics literature the matrices are finding numerous applications as well. Let us point out, for example, [14,15,16,17,18].

During the last decade, based on the seminal paper [19], important results

were obtained in the classification of gradings of classical simple Lie algebras [20,21,22]. For a given simple Lie algebra it is thus possible — with some effort — to determine all its gradings. The extreme gradings, which cannot be further refined ('fine gradings') are useful because other gradings are obtained from them by suitable combinations ('coarsening') of their grading subspaces. In mathematics, the fine gradings of simple Lie algebras are analogues of Cartan's root decomposition, which is one of them. They define new bases with uncommon properties. In physics, they provide maximal sets of quantum observables with additive quantum numbers.

The general goal of our efforts, which goes well beyond the present paper, is to find all the Lie algebras which can be obtained from $sl(3,\mathbb{C})$ by one–step contraction. That is, excluding from consideration chains of successive contractions for the same grading. There are four fine gradings of the algebra [23], i.e. gradings which cannot be further refined. Therefore our task splits into four smaller ones, one for each fine grading. All gradings of $sl(3,\mathbb{C})$ are shown on Fig. 1, including the four fine ones [24].

Best known of the four fine gradings [23] of $sl(3,\mathbb{C})$ is called the root decomposition or toroidal grading. There the algebra is decomposed as a linear space into the direct sum of eigenspaces of a maximal torus of the group $SL(3,\mathbb{C})$, or equivalently, into eigenspaces of the corresponding Cartan subalgebra. All gradings, arising in contractions preserving the toroidal grading, were found in [25].

The present case involves $sl(3,\mathbb{C})$ decomposed into eigenspaces of the adjoint action of generalized Pauli matrices. Although the role of these matrices in grading $sl(n,\mathbb{C})$ was singled out only recently [10], the special properties of the associative algebra of these matrices were known long before [13] (also [16] and references therein). Remaining two gradings both involve outer automorphisms of $sl(3,\mathbb{C})$. Corresponding contractions have not been studied so far.

Solving the four contraction problems of $sl(3,\mathbb{C})$ results in four lists of contracted algebras. Subsequently the lists need to be purged of overlaps. Relative practical difficulty of the contraction problems can be read of Fig. 1. The graph of the Figure shows successive refinements of gradings leading from the trivial one (whole $sl(3,\mathbb{C})$) to the four fine gradings. Nodes of the graph stand for non-equivalent gradings, links (arrows) indicate refinements. The graph exhibits 8 levels corresponding to numbers of grading subspaces: the numbers increase downwards, from 1 to 8, starting from the level of $sl(3,\mathbb{C})$ itself.

In order to see the relative difficulty, we split the graph into four overlapping subgraphs as follows. Retain in each of the subgraphs only the links and nodes which connect one fine grading with the whole $sl(3,\mathbb{C})$. A node which appears on several of the subgraphs indicates that the contracted Lie

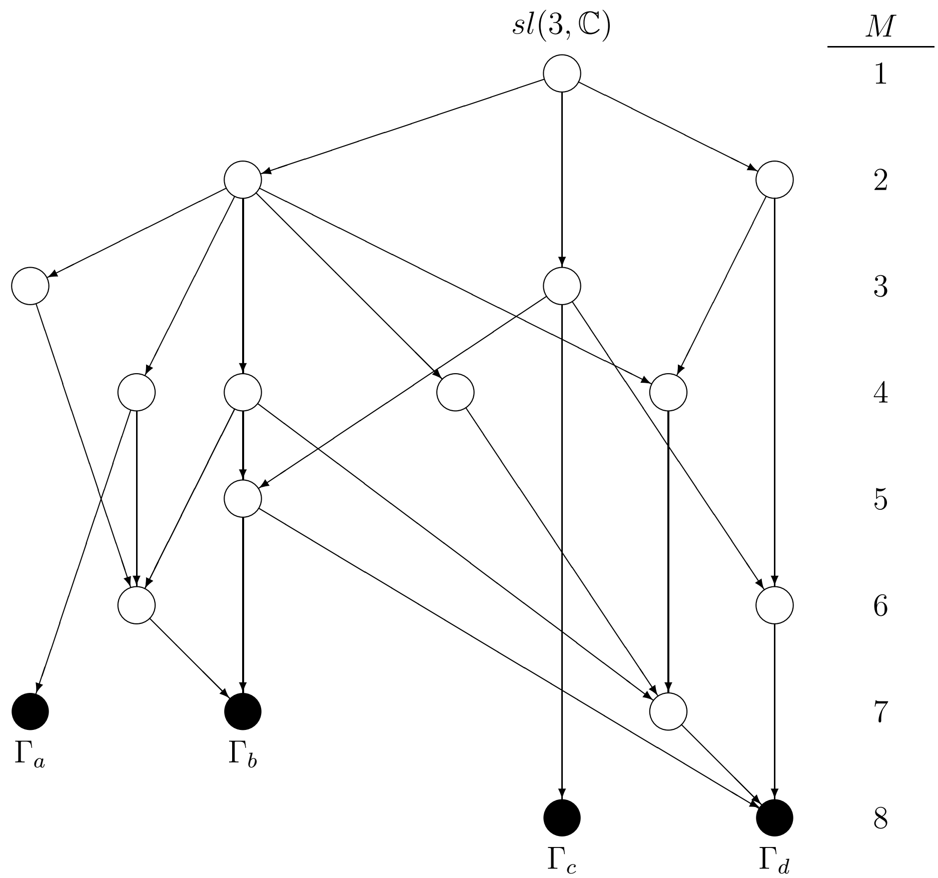

Fig. 1. The hierarchy of 17 gradings of $sl(3,\mathbb{C})$ [24]; Γ_b and Γ_c denote toroidal and Pauli gradings, respectively. The gradings are distributed into 8 levels according to the number M of their grading subspaces. Nodes of the graph stand for non-equivalent gradings, links (arrows) indicate refinements. Black circles denote fine gradings.

algebras, corresponding to the grading of that node, appear on several of the four lists. The less nodes a subgraph contains, the more complicated it is to find all contractions for that fine grading. In such a comparison, description of the contractions preserving the Pauli grading turns out to be a far more challenging problem than the other three cases.

Specific aim of this article is to provide the list of 188 sets of structure constants of the Lie algebras resulting from the contractions of $sl(3,\mathbb{C})$ which preserve the Pauli grading. On this basis, isomorphism classes of contracted algebras are determined.

A general new contribution of this paper in development of the graded contraction method, is systematic exploitation of the symmetries of the algebraic equations for the contraction parameters while solving the equations.

The goal of this paper is to identify the Lie algebras from the structure constants given by each contraction matrix. More precisely, we determine the subset of those which are pairwise non-isomorphic. The problem is particularly challenging for solvable Lie algebras which are indecomposable and whose dimension exceeds 5. Namely, little is known about these algebras, neither their isomorphism classes, nor even estimates of their number. Practically that means that existing algorithms for identification of Lie algebras had to be extended to allow us to recognize isomorphic pairs among Lie algebras arising from contractions. Here we shall restrict ourselves to (finite-dimensional) Lie algebras over complex numbers $\mathbb{C}$.

In the course of the work it turned out that finding an isomorphism between algebras was a very non-trivial question. We computed invariants for the algebras and when they were different, the algebras were clearly non-isomorphic, but when they were the same, new criteria had to be found. On the other hand we can try to find an isomorphism in an explicit way. Corresponding system of quadratic equations is generally more complicated than the system of contraction equations and, moreover, the symmetries for this system are not known. However, in some cases we were able to solve this system on computer.

Our main method for identifying a Lie algebra given by its structure constants was the paper by Rand, Winternitz and Zassenhaus [32].

Solutions of the system of contraction equations were written in the form of 8×8 contraction matrices ε with 24 relevant entries. We start from the 8–dimensional Lie algebra $sl(3,\mathbb{C})$ given by the structure constants $c_{i,j}^k$ in the basis of the Pauli grading $\{e_i\}_{i=1}^8$:

$$[e_i, e_j] = \sum_{k=1}^{8} c_{i,j}^k e_k, \qquad i,j = 1,\ldots,8. \tag{1}$$

Then the solution ε determines the contracted algebra with the structure constants (in the same basis) given by

$$c(\varepsilon)_{i,j}^k = \varepsilon_{i,j}\ c_{i,j}^k. \tag{2}$$

In Section 2, properties of gradings are briefly reviewed. The Pauli grading of $sl(3,\mathbb{C})$ is described explicitly, since it is the starting point of our calculations.

In Section 3, we review the notion of a graded contraction as a tool for obtaining new non-simple Pauli graded 8-dimensional Lie algebras. Contrast between generic two-term contraction equations and equations with three terms is underlined. A normalization process and the concepts of continuous and discrete graded contractions are introduced.

In Section 4, the notion of symmetry group of a grading is introduced, its

action on contraction matrices is defined, and it is shown that the action induces the symmetries of the system of contraction equations. Crucial definition of equivalence of solutions and the fact that Lie algebras obtained by graded contractions corresponding to equivalent solutions are isomorphic, is presented.

Section 5 contains the system of 48 two-term equations for the Pauli grading of $sl(3,\mathbb{C})$. We show how the symmetry group can effectively help in reducing the number of equations. Then all higher-order identities [26] of order 2 and 3 are used to distinguish continuous contractions from the discrete ones.

In Section 6, Theorem 7 is used to determine all solutions of the contraction equations. A list of nonequivalent cases is found in Appendix A.

In Section 7, seven steps of our algorithm used for identification are described. In Section 8, the algorithm is applied to all solutions of the contraction system. In order to identify the contracted Lie algebras uniquely, the set of invariants was supplemented in section 9 by dimensions of algebras of derivations. The last section, Concluding remarks, contains a number of comments, in particular there is an example of three-term contraction equations and a number of interesting problems is pointed out. In concluding remarks one finds a comparison between the results of this paper and results for the toroidal grading [25]. For completeness, the commutation relations of all non-isomorphic contracted Lie algebras as well as computed invariants are tabulated in Appendix B.

2 Pauli grading of $sl(3,\mathbb{C})$

Pauli gradings of $sl(n,\mathbb{C})$, for any $2 \leq n < \infty$, were described in [10]. In the case of $sl(3,\mathbb{C})$ it is one of the four fine gradings of that Lie algebra. The grading decomposes $sl(3,\mathbb{C})$ into eight 1-dimensional subspaces.

In the defining 3-dimensional representation, basis vectors/generators are 3×3

generalized Pauli matrices:

$$\begin{aligned}
sl(3,\mathbb{C}) &= L_{01} \oplus L_{02} \oplus L_{10} \oplus L_{20} \oplus L_{11} \oplus L_{22} \oplus L_{12} \oplus L_{21} \qquad (3)\\
&= \mathbb{C}Q \oplus \mathbb{C}Q^2 \oplus \mathbb{C}P \oplus \mathbb{C}P^2 \oplus \mathbb{C}PQ \oplus \mathbb{C}P^2Q^2 \oplus \mathbb{C}PQ^2 \oplus \mathbb{C}P^2Q\\
&= \mathbb{C}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \oplus\\
&\oplus \mathbb{C}\begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 & 0 & \omega \\ 1 & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 & 0 & \omega^2 \\ 1 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}
\end{aligned}$$

where $\omega = \exp(2\pi\mathrm{i}/3)$. Putting $L_{rs} := \{X_{rs}\}_{lin} \equiv \mathbb{C}X_{rs} = \mathbb{C}Q^rP^s$, we have the commutation relations,

$$[X_{rs}, X_{r's'}] = (\omega^{sr'} - \omega^{rs'})X_{r+r',s+s' \pmod 3} \,. \qquad (4)$$

The index set I for the Pauli grading consists of 8 couples rs, where $r, s = 0, 1, 2$ with the exception of 00.

Subsequently we make extensive use of the symmetry group of the Pauli grading. It was described in detail in [27] as a finite matrix group

$$H_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{Z}_3,\ ad - bc = \pm 1 \pmod 3 \right\}. \qquad (5)$$

It contains the subgroup of matrices with determinant $+1$ called $SL(2, \mathbb{Z}_3)$.

3 Graded contractions of $sl(3,\mathbb{C})$

3.1 Contraction parameters

The commutation relations (4) for basis elements X_{rs} are modified, for the purpose of a graded contraction [9] of the Lie algebra, by introduction of the contraction parameters ε,

$$\begin{aligned}
[X_{rs}, X_{r's'}]_\varepsilon :&= \varepsilon_{(rs)(r's')}[X_{rs}, X_{r's'}] \qquad (6)\\
&= \varepsilon_{(rs)(r's')}(\omega^{sr'} - \omega^{rs'})X_{r+r',s+s' \pmod 3} \,. \qquad (7)
\end{aligned}$$

Requirement, that the result of a contraction is a Lie algebra, imposes certain conditions on the contraction parameters. Antisymmetry of the modified commutator $[\ ,\ ]_\varepsilon$ immediately gives

$$\varepsilon_{(rs)(r's')} = \varepsilon_{(r's')(rs)}, \tag{8}$$

hence it is convenient to view the set of contraction parameters as a symmetric (8×8) **contraction matrix**. Among its 36 independent matrix elements, twelve are **irrelevant**, because in (6) they are multiplied by vanishing commutators of $sl(3, \mathbb{C})$. Consequently there are only 24 **relevant** contraction parameters. They are still subject to conditions imposed by the Jacobi identities, i.e.

$$e(i\ j\ k): \quad [x, [y, z]_\varepsilon]_\varepsilon + [z, [x, y]_\varepsilon]_\varepsilon + [y, [z, x]_\varepsilon]_\varepsilon = 0, \tag{9}$$

where $x \in L_i$, $y \in L_j$, $z \in L_k$, $i = (rs)$, $j = (r's')$, $k = (r''s'')$. Each $e(i\ j\ k)$ identifies a **contraction equation**. We call the set of contraction equations **contraction system** $\mathcal{S}$, and the set of its solutions is denoted $\mathcal{R}(\mathcal{S})$. Using the Jacobi identity in $sl(3, \mathbb{C})$,

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \tag{10}$$

one can rewrite (9) in the form

$$e(i\ j\ k): \quad (\varepsilon_{i,j+k}\varepsilon_{jk} - \varepsilon_{k,i+j}\varepsilon_{ij})[x, [y, z]] + (\varepsilon_{j,k+i}\varepsilon_{ki} - \varepsilon_{k,i+j}\varepsilon_{ij})[y, [z, x]] = 0.$$

In generic cases, when there exist $x' \in L_i$, $y' \in L_j$, $z' \in L_k$ such that $[x', [y', z']]$ and $[y', [z', x']]$ are linearly independent, then equation (9) is equivalent to 2 two-term equations

$$e(i\ j\ k): \quad \varepsilon_{i,j+k}\varepsilon_{jk} = \varepsilon_{k,i+j}\varepsilon_{ij} = \varepsilon_{j,k+i}\varepsilon_{ki}. \tag{11}$$

However, this condition is not always fulfilled. Then a three-term contraction equations arises. See Concluding remarks for an example.

3.2 Equivalence transformations

The system $\mathcal{S}$ of quadratic equations admits many solutions. Our task is to find its solutions which yield non-isomorphic Lie algebras $\mathcal{L}^\varepsilon$. It is a practical imperative to make use of transformations which leave $\mathcal{S}$ unchanged, but transforming otherwise the contraction parameters. In this subsection we introduce renormalization of parameters ε.

First we introduce, following [9], **commutative elementwise matrix multiplication** denoted by $\bullet$. For two matrices $A = (A_{ij})$, $B = (B_{ij})$ we define the matrix $C := (C_{ij})$ by the formula

$$C_{ij} := A_{ij}B_{ij} \qquad \text{(no summation)} \tag{12}$$

and write $C = A \bullet B$.

For a given grading we renormalize grading subspaces, according to

$$L_k \quad \longrightarrow \quad a_k L_k \,, \qquad k \in I, \quad 0 \neq a_k \in \mathbb{C}\,.$$

Here k takes value from an index set I of the grading. Matrix $\alpha := (\alpha_{ij})$, where

$$\alpha_{ij} = \frac{a_i a_j}{a_{i+j}} \quad \text{for } i,j \in I\,, \tag{13}$$

is a **normalization matrix**.

Normalization is a process based on the following lemma:

Lemma 1 *Let $\mathcal{L}^\varepsilon$ be a graded contraction of a graded Lie algebra $\mathcal{L} = \bigoplus_{i \in I} L_i$. Then $\mathcal{L}^\mu$, where $\mu = \alpha \bullet \varepsilon$, is for any normalization matrix α a graded contraction of $\mathcal{L}$ and the Lie algebras $\mathcal{L}^\mu$ and $\mathcal{L}^\varepsilon$ are isomorphic.*

In many cases it is possible, by a suitable choice of the constants a_k, $k \in I$, to transform matrix elements of ε to 1's and 0's only. We conclude in *Example* 8 (Sec. 6.2). that solution for the Pauli grading of $sl(3, \mathbb{C})$ which has all contraction parameters non-vanishing can be always normalized to solution with 1's only. Clearly, this fact is equivalent to:

Proposition 2 *Every contraction matrix of the Pauli graded $sl(3, \mathbb{C})$ without zeros on relevant positions can be written in a form of normalization matrix (13).*

3.3 Continuous and discrete graded contractions

There are solutions of two types of a given contraction system $\mathcal{S}$, continuous and discrete ones.

A solution $\varepsilon \in \mathcal{R}(\mathcal{S})$ is **continuous** if there exists a continuous set of solutions $\varepsilon(t) \in \mathcal{R}(\mathcal{S})$, $0 < t \leq 1$, such that, for all relevant contraction parameters, one has

$$\varepsilon_{ij}(1) = 1, \qquad \varepsilon_{ij}(t) \neq 0, \qquad \varepsilon_{ij} = \lim_{t \to 0} \varepsilon_{ij}(t). \tag{14}$$

If a solution is not continuous, then it is called **discrete**.

If, for a given continuous graded contraction $\varepsilon \in \mathcal{R}(\mathcal{S})$, the corresponding continuous set of solutions has the form

$$\varepsilon_{ij}(t) = t^{n_i+n_j-n_{i+j}} \quad i,j \in I, \quad n_i, n_j, n_{i+j} \in \mathbb{Z}, \tag{15}$$

then ε is a generalized Inönü-Wigner contraction [26].

Traditionally continuous contractions were more thoroughly studied in the literature [26]. Let us point out higher-order identities which we use in Sect. 5.1 as an effective tool for identifying *discrete* graded contractions in the present case.

4 Symmetries and graded contractions

The contraction system $\mathcal{S}$ for the Pauli grading of $sl(3,\mathbb{C})$ is the system of 48 quadratic equations in 24 variables. Making use of the symmetries should simplify the solution of such a large system.

Symmetries are involved in three different ways in our problem. First it is the symmetry group of the Pauli grading of $sl(3,\mathbb{C})$. Second is the symmetry of the system $\mathcal{S}$ of contraction equations. And third, the symmetries transforming solutions among themselves.

4.1 Symmetry group of the Pauli grading

For general gradings, symmetry groups were introduced in [19] and, for Pauli gradings of $sl(n,\mathbb{C})$ were described in [27]. We recall in particular the following:

The symmetry group $\mathrm{Aut}\,\Gamma$ of a grading $\Gamma : \mathcal{L} = \bigoplus_{i\in I} L_i$ is defined as a subgroup of $\mathrm{Aut}\,\mathcal{L}$ which contains automorphisms g with the property $gL_i = L_{\pi_g(i)}$, where $\pi_g : I \to I$ is a permutation of the index set I.

This group is described in detail in [27,28,23], where an important theorem was proved; we specify it for $n = 3$:

Theorem 3 *The symmetry group of the Pauli grading of $sl(3,\mathbb{C})$ is isomor-*

phic to the matrix group H_3. It is a finite matrix group

$$H_3 = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{Z}_3,\ ad - bc = \pm 1 \pmod 3 \right\}. \tag{16}$$

Let π_A denote the permutation corresponding to a matrix $A \in H_3$; then the action of π_A on the grading indices is given by

$$\pi_A\,(i\ j) = (i\ j) \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{17}$$

where matrix multiplication modulo 3 is understood.

Note that H_3 contains the subgroup $SL(2, \mathbb{Z}_3)$ of matrices with determinant $+1$.

4.2 Action of the symmetry group of the grading

Let us define a set of **relevant pairs of grading indices** $\mathcal{I}$ by

$$\mathcal{I} := \left\{ i\,j \,\middle|\, i, j \in I,\ [L_i, L_j] \neq \{0\} \right\}. \tag{18}$$

For the Pauli grading of $sl(3, \mathbb{C})$ we obtain explicitly

$$\mathcal{I} = \left\{ (ij)(kl) \,\middle|\, jk - il \neq 0 \,(\mathrm{mod}\ 3),\ (ij), (kl) \in \mathbb{Z}_3 \times \mathbb{Z}_3 \backslash \{(0,0)\} \right\} \tag{19}$$

by analyzing relations (4). A set of **relevant contraction parameters** ε_{ij}, due to (8), can be written as $\mathcal{E} := \{\varepsilon_k,\ k \in \mathcal{I}\}$. For a permutation π and a contraction matrix $\varepsilon = (\varepsilon_{ij})$, an **action of π on a contraction matrix** $\varepsilon \mapsto \varepsilon^\pi$ is defined by

$$(\varepsilon^\pi)_{ij} := \varepsilon_{\pi(i)\pi(j)}. \tag{20}$$

We observe that an **action on variables** $\varepsilon_{ij} \mapsto \varepsilon_{\pi(i)\pi(j)}$ is in fact the action on the set of relevant variables $\mathcal{E}$: if $\varepsilon_{ij} \in \mathcal{E}$, $[L_i, L_j] \neq \{0\}$ and $g \in \mathrm{Aut}\,\Gamma$, π the corresponding permutation, then $\{0\} \neq g[L_i, L_j] = [L_{\pi(i)}, L_{\pi(j)}]$ and $\varepsilon_{\pi(i)\pi(j)} \in \mathcal{E}$. Hence even if the matrix ε^π has, in general, zeros on different positions than matrix ε, the matrix ε^π has zeros on the same irrelevant positions as the matrix ε, i.e. the irrelevant positions are fixed.

We verify that (20) is a well-defined action of the group $\mathrm{Aut}\,\Gamma$ on the set $\mathcal{R}(\mathcal{S})$ of solutions of the contraction system. However, in Subsect. 4.4 we are going

to widen the definition of equivalence on the solutions. The following lemma shows that the action of π induces an isomorphism of Lie algebras associated with contraction matrices ε and ε^π.

Lemma 4 *Let $\mathcal{L}^\varepsilon$ be a graded contraction of a Γ–graded Lie algebra $\mathcal{L} = \bigoplus_{i\in I} L_i$. Then $\mathcal{L}^{\varepsilon^\pi}$ is, for permutations π corresponding to elements of* $\operatorname{Aut}\Gamma$*, also a graded contraction of $\mathcal{L}$ and the Lie algebras $\mathcal{L}^{\varepsilon^\pi}$ and $\mathcal{L}^\varepsilon$ are isomorphic, $\mathcal{L}^{\varepsilon^\pi} \simeq \mathcal{L}^\varepsilon$.*

PROOF. For $g \in \operatorname{Aut}\Gamma$ take the corresponding π. Consider

$$gx = z,\ gy = w, \quad x \in L_i,\ y \in L_j,\ z \in L_{\pi(i)},\ w \in L_{\pi(j)},\ i, j \in I. \tag{21}$$

Then for all x, y the bilinear mapping $[x, y]_{\varepsilon^\pi} = \varepsilon_{\pi(i)\pi(j)}[x, y]$ and the Lie bracket $[x, y]_\varepsilon = \varepsilon_{ij}[x, y]$ satisfy

$$[x, y]_{\varepsilon^\pi} = \varepsilon_{\pi(i)\pi(j)}[x, y] = \varepsilon_{\pi(i)\pi(j)}g^{-1}[z, w] = g^{-1}[gx, gy]_\varepsilon. \tag{22}$$

Hence $\mathcal{L}^{\varepsilon^\pi}$ is a Lie algebra and g is an isomorphism between $\mathcal{L}^{\varepsilon^\pi}$ and $\mathcal{L}^\varepsilon$.

4.3 Symmetries of the system of contraction equations

It follows from Lemma 4 that for a given solution ε one can construct new contraction matrices ε^π which are again solutions of the contraction system (and correspond to isomorphic Lie algebras). We thus obtained the substitutions $\varepsilon_{ij} \mapsto \varepsilon_{\pi(i)\pi(j)}$ preserving the set of solutions of the contraction system.

Now we can define an **action of** $\operatorname{Aut}\Gamma$ **on the contraction system** $\mathcal{S}$: each equation in $\mathcal{S}$ is labeled by a triple of grading indices and we write $e(i\,j\,k) \in \mathcal{S}$ in the form

$$e(i\ j\ k) : [x, [y, z]_\varepsilon]_\varepsilon + \text{cyclically} = 0, \quad x \in L_i, y \in L_j, z \in L_k; \tag{23}$$

then for each π we define the action

$$e(i\ j\ k) \mapsto e(\pi(i)\ \pi(j)\ \pi(k)). \tag{24}$$

Note that equation $e(\pi(i)\,\pi(j)\,\pi(k))$ can be written for each g and the corresponding π as

$$e(\pi(i)\ \pi(j)\ \pi(k)) : [gx, [gy, gz]_\varepsilon]_\varepsilon + \text{cyclically} = 0. \tag{25}$$

According to (22) this is equal to

$$g[x,[y,z]_{\varepsilon^\pi}]_{\varepsilon^\pi} + \text{cyclically} = 0 \tag{26}$$

and (26) is satisfied if and only if

$$[x,[y,z]_{\varepsilon^\pi}]_{\varepsilon^\pi} + \text{cyclically} = 0. \tag{27}$$

Equation (27) is precisely the equation (23) after the substitution $\varepsilon_{ij} \mapsto \varepsilon_{\pi(i)\pi(j)}$. In this way we have not only verified the invariance of the contraction system (up to equivalence of equations), but also have shown the method which is subsequently used for its construction. Namely, having chosen a starting equation one can write a whole orbit of equations merely by substituting $\varepsilon_{ij} \mapsto \varepsilon_{\pi(i)\pi(j)}$ till all permutations π corresponding to $\operatorname{Aut}\Gamma$ are exhausted.

If we denote unordered k-tuple of grading indices $i_1, i_2, \ldots, i_k \in I$ as $i_1\, i_2 \ldots i_k$ and for permutation π corresponding to $g \in \operatorname{Aut}\Gamma$ define an **action on unordered k-tuples** as

$$i_1\, i_2\, \ldots i_k \;\mapsto\; \pi(i_1)\, \pi(i_2) \ldots \pi(i_k), \tag{28}$$

then it is clear that orbits of equations correspond to orbits of unordered triples of grading indices. In the following, we shall write also $\pi \in \operatorname{Aut}\Gamma$ for a permutation corresponding to some $g \in \operatorname{Aut}\Gamma$.

4.4 Equivalence of solutions

Combining Lemma 1 and Lemma 4 it is easy to see that an equivalence relation on the set $\mathcal{R}(\mathcal{S})$ naturally arises: we define two solutions $\varepsilon', \varepsilon \in \mathcal{R}(\mathcal{S})$ to be **equivalent**, $\varepsilon' \sim \varepsilon$, if there exists a normalization matrix α and a permutation π (corresponding to an element of $\operatorname{Aut}\Gamma$) such that

$$\varepsilon' = \alpha \bullet \varepsilon^\pi. \tag{29}$$

Note that this definition of equivalence is different from that of [26], Definition 2.2. For instance it is clear that two equivalent solutions *do not* have the same set of zeros. It follows from Lemma 1 and Lemma 4 that the following Proposition holds:

Proposition 5 *Lie algebras corresponding to equivalent solutions of the system of contraction equations are isomorphic.*

5 Contraction system for the Pauli grading of $sl(3,\mathbb{C})$

In this section we describe the system $\mathcal{S}$ of 48 quadratic contraction equations for the Pauli grading of $sl(3,\mathbb{C})$ using the symmetry group.

Let us take the Pauli grading in the form (3), the grading group is $\mathbb{Z}_3\times\mathbb{Z}_3$; no subspace is labeled by $(0,0)$. We choose the explicit form of matrix ε defined by (6); it is an 8×8 symmetric matrix with 24 relevant variables

$$\varepsilon = \begin{pmatrix} 0 & 0 & \varepsilon_{(01)(10)} & \varepsilon_{(01)(20)} & \varepsilon_{(01)(11)} & \varepsilon_{(01)(22)} & \varepsilon_{(01)(12)} & \varepsilon_{(01)(21)} \\ 0 & 0 & \varepsilon_{(02)(10)} & \varepsilon_{(02)(20)} & \varepsilon_{(02)(11)} & \varepsilon_{(02)(22)} & \varepsilon_{(02)(12)} & \varepsilon_{(02)(21)} \\ \varepsilon_{(01)(10)} & \varepsilon_{(02)(10)} & 0 & 0 & \varepsilon_{(10)(11)} & \varepsilon_{(10)(22)} & \varepsilon_{(10)(12)} & \varepsilon_{(10)(21)} \\ \varepsilon_{(01)(20)} & \varepsilon_{(02)(20)} & 0 & 0 & \varepsilon_{(20)(11)} & \varepsilon_{(20)(22)} & \varepsilon_{(20)(12)} & \varepsilon_{(20)(21)} \\ \varepsilon_{(01)(11)} & \varepsilon_{(02)(11)} & \varepsilon_{(10)(11)} & \varepsilon_{(20)(11)} & 0 & 0 & \varepsilon_{(11)(12)} & \varepsilon_{(11)(21)} \\ \varepsilon_{(01)(22)} & \varepsilon_{(02)(22)} & \varepsilon_{(10)(22)} & \varepsilon_{(20)(22)} & 0 & 0 & \varepsilon_{(22)(12)} & \varepsilon_{(22)(21)} \\ \varepsilon_{(01)(12)} & \varepsilon_{(02)(12)} & \varepsilon_{(10)(12)} & \varepsilon_{(20)(12)} & \varepsilon_{(11)(12)} & \varepsilon_{(22)(12)} & 0 & 0 \\ \varepsilon_{(01)(21)} & \varepsilon_{(02)(21)} & \varepsilon_{(10)(21)} & \varepsilon_{(20)(21)} & \varepsilon_{(11)(21)} & \varepsilon_{(22)(21)} & 0 & 0 \end{pmatrix}. \quad (30)$$

Contraction equations $e((ij)\,(kl)\,(mn))\in\mathcal{S}$ should hold for all possible triples of indices $(ij),(kl),(mn)$. It is clear that for each triple, whose two indices are identical, the equation is identically fulfilled. Equations also do not depend on the ordering of the triples. The number of equations is then equal to the combination number $\binom{8}{3}=56$. Moreover, equations with $i+k+m=0$ and $j+l+n=0$ simultaneously are also fulfilled identically. This situation arises in eight cases. Hence the contraction system consists of 48 equations.

Let us show an example of an equation computed for a chosen triple, say $(01)(02)(10)$. The result is

$$[\varepsilon_{(02)(10)}\varepsilon_{(01)(12)}(\omega^2-1)(\omega-1)+0+\varepsilon_{(10)(01)}\varepsilon_{(02)(11)}(1-\omega)(\omega^2-1)]X_{10}=0. \quad (31)$$

Recall that the symmetry group H_3 has 48 elements. It turns out that its application on the triples leads to exactly *two* orbits of its action, each consisting of 24 distinct triplets. We choose the triples $(01)(02)(10)$ and $(01)(10)(11)$ as representative elements of the two orbits. We observe that all 24 elements of each orbit are obtained by the action of 24 elements of the subgroup $SL(2,\mathbb{Z}_3)\subset H_3$ starting from an arbitrary point. Then for our choice of representative points,

our system $\mathcal{S}$ can be written simply as

$$\mathcal{S}^a \ : \varepsilon_{(02)(10)A}\varepsilon_{(01)(12)A} - \varepsilon_{(01)(10)A}\varepsilon_{(02)(11)A} = 0 \tag{32}$$

$$\mathcal{S}^b \ : \varepsilon_{(10)(11)A}\varepsilon_{(01)(21)A} - \varepsilon_{(01)(11)A}\varepsilon_{(10)(12)A} = 0 \qquad \forall A \in SL(2, \mathbb{Z}_3) \tag{33}$$

where we used the abbreviation $\varepsilon_{(ij)(kl)A} := \varepsilon_{(ij)A,(kl)A}$.

It turns out that the system $\mathcal{S}^a$ contains dependent equations which can be eliminated. This is seen as follows. Equation obtained from (32) by the 'action' of the unit matrix can also be written in the form

$$\varepsilon_{(01)(10)X}\varepsilon_{(02)(11)X} - \varepsilon_{(01)(10)}\varepsilon_{(02)(11)} = 0, \qquad X = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \tag{34}$$

This is due to the fact that the quadruples of indices [(02)(10)][(01)(12)] and [(01)(10)][(02)(11)] lie in the same $SL(2, \mathbb{Z}_3)$–orbit (the pairs of indices in brackets [,] and the pairs of these brackets are unordered). The equation generated from equation (34) by the matrix $A = X$

$$\varepsilon_{(01)(10)X^2}\varepsilon_{(02)(11)X^2} - \varepsilon_{(01)(10)X}\varepsilon_{(02)(11)X} = 0 \tag{35}$$

is also contained in $\mathcal{S}^a$, due to (32). By adding equations (34) and (35) we get

$$\varepsilon_{(01)(10)X^2}\varepsilon_{(02)(11)X^2} - \varepsilon_{(01)(10)}\varepsilon_{(02)(11)} = 0. \tag{36}$$

Since $X^3 = 1$ holds, equation (36) is generated from equation (32) by matrix $A = X^2$. Hence we conclude that the left cosets of $SL(2, \mathbb{Z}_3)$ with respect to the cyclic subgroup $\{1, X, X^2\}$ generate the triples of dependent equations. By Lagrange's theorem, the number of these cosets is $24/3 = 8$. In this way we obtained 8 equations (one to each coset) which can be eliminated from the system $\mathcal{S}^a$. Concerning $\mathcal{S}^b$, we observe that the quadruples of indices [(10)(11)][(01)(21)] and [(01)(11)][(10)(12)] *do not* lie in the same orbit. Therefore the equations of $\mathcal{S}^b$ are independent.

5.1 Higher–order identities

Higher–order identities will be useful for distinguishing between continuous and discrete contractions. A **higher–order identity of order** k is defined [26] as an equation of the type

$$\varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_k} = \varepsilon_{j_1}\varepsilon_{j_2}\cdots\varepsilon_{j_k}; \tag{37}$$

here $k \in \mathbb{N}$, $i_1, i_2 \cdots i_k$ and $j_1, j_2 \cdots j_k$ are disjoint sets of relevant pairs of grading indices, and equation (37) holds for all contraction matrices ε without zeros on all relevant positions (i.e., $\varepsilon_i \neq 0$ for all $i \in \mathcal{I}$), but is violated by

some contraction matrix with zero on some relevant position. It is obvious that continuous solutions, as limits of non-zero solutions, do have to satisfy all higher–order identities (see also Theorem 5.1 [26]):

Proposition 6 *Let a solution $\varepsilon \in \mathcal{R}(\mathcal{S})$ of the contraction system $\mathcal{S}$ be a continuous graded contraction. Then ε satisfies all higher–order identities.*

It follows that those solutions which violate any higher–order identity are discrete.

5.2 Higher–order identities for the Pauli grading of $sl(3,\mathbb{C})$

As an example of a third order identity for the Pauli grading one can give the following equation:

$$\varepsilon_{(01)(10)}\varepsilon_{(01)(12)}\varepsilon_{(02)(21)} = \varepsilon_{(01)(22)}\varepsilon_{(01)(21)}\varepsilon_{(02)(12)}. \tag{38}$$

It can either be deduced directly from the system $\mathcal{S}$, or we can use the fact (Proposition 2) that all solutions with all non-zero relevant elements can be written in the form (13) of the normalization matrix α. Hence the identity

$$\frac{a_{(01)}a_{(10)}}{a_{(11)}}\frac{a_{(01)}a_{(12)}}{a_{(10)}}\frac{a_{(02)}a_{(21)}}{a_{(20)}} = \frac{a_{(01)}a_{(22)}}{a_{(20)}}\frac{a_{(01)}a_{(21)}}{a_{(22)}}\frac{a_{(02)}a_{(12)}}{a_{(11)}} \tag{39}$$

is evidently satisfied. We observe that the identity (38) is evidently violated for instance by the contraction matrix equivalent to $\varepsilon_{21,4}$ given in Appendix: $\varepsilon_{(01)(10)} = \varepsilon_{(01)(12)} = \varepsilon_{(02)(21)} = 1$ and the other ε's are equal to zero.

Applying the symmetry group H_3 to (38), we can write the 24-point orbit of higher-order identities in the form

$$\varepsilon_{(01)(10)A}\varepsilon_{(01)(12)A}\varepsilon_{(02)(21)A} = \varepsilon_{(01)(22)A}\varepsilon_{(01)(21)A}\varepsilon_{(02)(12)A} \qquad \forall A \in H_3 \tag{40}$$

Note that the action is effective only for 24 elements of the subgroup $SL(2,\mathbb{Z}_3)$.

We have found a set of second and third order identities. In all we have 104 such identities, 24 of them being of second order. Table 1 lists their representative points and the numbers of the resulting identities under the action of $SL(2,\mathbb{Z}_3)$.

For each solution of the system $\mathcal{S}$ we were able to decide that one of two exclusive alternatives holds:

- either we found that a solution violates some of 104 identities listed in Table 1 and therefore it is *discrete*
- or we explicitly found a continuous path of the form (15), hence the solution is *continuous*

Thus is was not necessary to investigate the completeness of our set of higher-order identities.

Order	Representative equation	Number of equations
2	$\varepsilon_{(01)(10)}\varepsilon_{(02)(11)} = \varepsilon_{(01)(20)}\varepsilon_{(02)(21)}$	*24*
3	$\varepsilon_{(01)(10)}\varepsilon_{(01)(11)}\varepsilon_{(01)(12)} = \varepsilon_{(01)(20)}\varepsilon_{(01)(22)}\varepsilon_{(01)(21)}$	*8*
3	$\varepsilon_{(01)(10)}\varepsilon_{(01)(12)}\varepsilon_{(02)(21)} = \varepsilon_{(01)(22)}\varepsilon_{(01)(21)}\varepsilon_{(02)(12)}$	*24*
3	$\varepsilon_{(01)(10)}\varepsilon_{(01)(11)}\varepsilon_{(02)(21)} = \varepsilon_{(01)(22)}\varepsilon_{(01)(21)}\varepsilon_{(02)(10)}$	*24*
3	$\varepsilon_{(01)(10)}\varepsilon_{(01)(11)}\varepsilon_{(02)(22)} = \varepsilon_{(01)(20)}\varepsilon_{(01)(22)}\varepsilon_{(02)(10)}$	*24*

Table 1
Orbits of 2nd and 3rd order identities for the Pauli grading

6 Solution of the system $\mathcal{S}$ of contraction equations for the Pauli grading of $sl(3,\mathbb{C})$

The goal of this section is to determine all equivalence classes (in the sense of Sect. 4.4) of solutions of the nonlinear contraction system. Since the method of solution proposed by [29] was inapplicable in our case, we had to find another way. Of course, a laborious case by case analysis as in [25] is always possible. However, we describe a method simplifying this laborious analysis. Our approach takes advantage of the symmetries of the Pauli grading, more precisely, the symmetries of the system of contraction equations as well as the symmetries defining equivalence classes of solutions. We obtained the complete set of (equivalence classes of) solutions, among them 13 which depend on one or two parameters.

6.1 The algorithm for solving the system $\mathcal{S}$

Our method is based on the fact that under suitable assumptions, the system $\mathcal{S}$ can be easily explicitly solved. But after leaving the assumption, i.e. putting

zero on the position we had assumed non–zero before, the solving becomes far more complicated. We shall formulate an algorithm allowing us to bypass this problem. It is based on the following theorem employing our notion of equivalent solutions (29).

Theorem 7 *Let $\mathcal{R}(\mathcal{S})$ be the set of solutions and $\mathcal{I}$ the set of relevant pairs of unordered indices of the contraction system $\mathcal{S}$ of a graded Lie algebra $\mathcal{L} = \bigoplus_{i\in I} L_i$. For any subsets $\mathcal{Q} \subset \mathcal{R}(\mathcal{S})$ and $\mathcal{P} = \{k_1, k_2, \ldots, k_m\} \subset \mathcal{I}$ we denote*

$$\mathcal{R}_0 := \left\{\varepsilon \in \mathcal{R}(\mathcal{S}) \middle| (\forall \varepsilon' \in \mathcal{Q})(\varepsilon \nsim \varepsilon')\right\}$$
$$\mathcal{R}_1 := \left\{\varepsilon \in \mathcal{R}_0 \,\middle|\, (\forall k \in \mathcal{P})(\varepsilon_k \neq 0)\right\}.$$

Then a solution $\varepsilon \in \mathcal{R}_0$ is not equivalent to any solution in $\mathcal{R}_1$ if and only if the following system of equations holds:

$$\begin{aligned} \varepsilon_{\pi_1(k_1)}\varepsilon_{\pi_1(k_2)}\cdots\varepsilon_{\pi_1(k_m)} &= 0 \\ &\vdots \\ \varepsilon_{\pi_n(k_1)}\varepsilon_{\pi_n(k_2)}\cdots\varepsilon_{\pi_n(k_m)} &= 0; \end{aligned} \tag{41}$$

here the set of permutations $\{\pi_1, \pi_2, \ldots, \pi_n\}$ exhausts all elements of the symmetry group $\mathrm{Aut}\,\Gamma$ of the grading.

PROOF. For any $\varepsilon \in \mathcal{R}_0$ we have (see 29):

$$(\exists \varepsilon' \in \mathcal{R}_1)(\varepsilon \sim \varepsilon') \Leftrightarrow (\exists \varepsilon' \in \mathcal{R}_1)(\exists \alpha)(\exists \pi \in \mathrm{Aut}\,\Gamma)(\alpha \bullet \varepsilon^\pi = \varepsilon') \tag{42}$$

$$\Leftrightarrow (\exists \alpha)(\exists \pi \in \mathrm{Aut}\,\Gamma)(\alpha \bullet \varepsilon^\pi \in \mathcal{R}_0 \text{ and } (\alpha \bullet \varepsilon^\pi)_k \neq 0,\ \forall k \in \mathcal{P}) \tag{43}$$

$$\Leftrightarrow (\exists \pi \in \mathrm{Aut}\,\Gamma)(\forall k \in \mathcal{P})((\varepsilon^\pi)_k \neq 0). \tag{44}$$

The equivalence (42) directly follows from the definition (29), the equivalence (43) expresses the trivial fact that $(\exists \varepsilon' \in \mathcal{R}_1)(\alpha \bullet \varepsilon^\pi = \varepsilon') \Leftrightarrow (\alpha \bullet \varepsilon^\pi \in \mathcal{R}_1)$. Since $\alpha \bullet \varepsilon^\pi \in \mathcal{R}_0$ is for any $\varepsilon \in \mathcal{R}_0$ automatically fulfilled and $(\alpha \bullet \varepsilon^\pi)_k \neq 0 \Leftrightarrow (\varepsilon^\pi)_k \neq 0$, the equivalence (44) follows.

Negating (44) we obtain

$$(\forall \varepsilon' \in \mathcal{R}_1)(\varepsilon \nsim \varepsilon') \Leftrightarrow (\forall \pi \in \mathrm{Aut}\,\Gamma)(\exists k \in \mathcal{P})((\varepsilon^\pi)_k = 0)$$

and this is the statement of the theorem.

We call the system of equations (41) corresponding to the sets $\mathcal{Q} \subset \mathcal{R}(\mathcal{S})$ and $\mathcal{P} \subset \mathcal{I}$ a **non-equivalence system**.

Repeated use of the theorem leads to the following algorithm for the evaluation of solutions:

(1) we set $\mathcal{Q} = \emptyset$ and suppose we have a set of assumptions $\mathcal{P}^0 \subset \mathcal{I}$. Then $\mathcal{R}_0 = \mathcal{R}(\mathcal{S})$, and we explicitly evaluate

$$\mathcal{R}^0 = \Big\{\varepsilon \in \mathcal{R}(\mathcal{S}) \mid (\forall k \in \mathcal{P}^0)(\varepsilon_k \neq 0)\Big\}$$

and write the *non-equivalence system* $\mathcal{S}^0$ of equations (41) corresponding to $\mathcal{Q} = \emptyset$, $\mathcal{P}^0$.

(2) we set $\mathcal{Q} = \mathcal{R}^0$ and suppose we have the set $\mathcal{P}^1 \subset \mathcal{I}$. Then $\mathcal{R}_0 = \mathcal{R}(\mathcal{S} \cup \mathcal{S}^0)$; furthermore we explicitly evaluate

$$\mathcal{R}^1 = \Big\{\varepsilon \in \mathcal{R}(\mathcal{S} \cup \mathcal{S}^0) \mid (\forall k \in \mathcal{P}^1)(\varepsilon_k \neq 0)\Big\}$$

and write the non-equivalence system $\mathcal{S}^1$ corresponding to $\mathcal{Q} = \mathcal{R}^0$, $\mathcal{P}^1$.

(3) we set $\mathcal{Q} = \mathcal{R}^0 \cup \mathcal{R}^1$. Then $\mathcal{R}_0 = \mathcal{R}(\mathcal{S} \cup \mathcal{S}^0 \cup \mathcal{S}^1)$ and we continue till we have evaluated the whole $\mathcal{R}(\mathcal{S})$ up to equivalence, i.e. till we have arrived at such $\mathcal{Q}$ that the corresponding set $\mathcal{R}_0$ is empty or trivial. Thus the idea is to repeatedly enlarge the set $\mathcal{Q}$ for as long as possible.

It is clear that the algorithm crucially depends on the choice of the subsets $\mathcal{P}^0, \mathcal{P}^1, \ldots$ of $\mathcal{I}$. Since the system $\mathcal{S}$ can be solved explicitly under the assumption that two of its variables do not vanish, we develop a theory for pairs from $\mathcal{I}$. For fixed $k \in \mathcal{I}$ we define an equivalence relation $\equiv^k$ on the set $\mathcal{I}^k := \mathcal{I} \setminus \{k\}$:
for $i, j \in \mathcal{I}^k$

$$i \equiv^k j \quad \Leftrightarrow \quad (\exists \pi \in \operatorname{Aut}\Gamma)(\pi\,(i\ k) = (j\ k)), \tag{45}$$

where $(i\ j)$ denotes an (unordered) pair $i, j \in \mathcal{I}$ and $\pi\,(i\ k) := (\pi(i)\ \pi(k))$.

The application of this equivalence will be seen in our concrete evaluation in the next subsection. We will make use of the following fact: the set of relevant indices $\mathcal{I}$ has 24 elements which are explicitly written in the matrix (30). We choose the index $k = (01)(10)$ and in Table 2 we list nine equivalence classes $\mathcal{I}_1^k, \ldots, \mathcal{I}_9^k$ of the equivalence $\equiv^k$:

6.2 Finding solutions of $\mathcal{S}$

The solutions are found in five consecutive steps. In each of the following steps, $k = (01)(10)$ is fixed, and it is assumed that the corresponding $\varepsilon_k \neq 0$. Let in $\mathcal{R}^m$ further $\varepsilon_i \neq 0$ be assumed. Then at the next step, in evaluating $\mathcal{R}^{m+1}$, one finds the following: the non–equivalence system $\mathcal{S}^m$ and previous assumption

$\mathcal{I}_1^k$	(11)(12), (11)(21), (22)(12), (22)(21)
$\mathcal{I}_2^k$	(01)(11), (10)(11), (01)(12), (10)(21)
$\mathcal{I}_3^k$	(02)(22), (20)(22), (02)(21), (20)(12)
$\mathcal{I}_4^k$	(01)(20), (02)(10)
$\mathcal{I}_5^k$	(01)(22), (10)(22)
$\mathcal{I}_6^k$	(01)(21), (10)(12)
$\mathcal{I}_7^k$	(02)(11), (20)(11)
$\mathcal{I}_8^k$	(02)(12), (20)(21)
$\mathcal{I}_9^k$	(02)(20)

Table 2
The equivalence classes of $\equiv^{(01)(10)}$

$\varepsilon_k \neq 0$ imply zeros on all positions j, $j \equiv^k i$. It is advantageous to take pairs of indices from the three largest classes in Table 2, namely, $(22)(21) \in \mathcal{I}_1^k$, $(10)(11) \in \mathcal{I}_2^k$, $(02)(22) \in \mathcal{I}_3^k$, in order to evaluate the sets $\mathcal{R}^0$, $\mathcal{R}^1$ and $\mathcal{R}^2$, $\mathcal{R}^3$ respectively.

We first list the five steps and then make more detailed explanations.

1. $\mathcal{R}^0 = \Big\{\varepsilon \in \mathcal{R}(\mathcal{S}) \mid \varepsilon_{(01)(10)} \neq 0,\, \varepsilon_{(22)(21)} \neq 0\Big\}$
   $\mathcal{S}^0:\ \varepsilon_{(01)(10)A}\varepsilon_{(22)(21)A} = 0 \quad \forall A \in H_3$
2. $\mathcal{R}^1 = \Big\{\varepsilon \in \mathcal{R}(\mathcal{S} \cup \mathcal{S}^0) \mid \varepsilon_{(01)(10)} \neq 0,\, \varepsilon_{(10)(11)} \neq 0,\, \varepsilon_{(01)(22)} \neq 0\Big\}$
   $\mathcal{S}^1:\ \varepsilon_{(01)(10)A}\varepsilon_{(10)(11)A}\varepsilon_{(01)(22)A} = 0 \quad \forall A \in H_3$
3. $\mathcal{R}^2 = \Big\{\varepsilon \in \mathcal{R}(\mathcal{S} \cup \mathcal{S}^0 \cup \mathcal{S}^1) \mid \varepsilon_{(01)(10)} \neq 0,\, \varepsilon_{(10)(11)} \neq 0\Big\}$
   $\mathcal{S}^2:\ \varepsilon_{(01)(10)A}\varepsilon_{(10)(11)A} = 0 \quad \forall A \in H_3$
4. $\mathcal{R}^3 = \Big\{\varepsilon \in \mathcal{R}(\mathcal{S} \cup \mathcal{S}^0 \cup \mathcal{S}^1 \cup \mathcal{S}^2) \mid \varepsilon_{(01)(10)} \neq 0,\, \varepsilon_{(02)(22)} \neq 0\Big\}$
   $\mathcal{S}^3:\ \varepsilon_{(01)(10)A}\varepsilon_{(02)(22)A} = 0 \quad \forall A \in H_3$
5. $\mathcal{R}^4 = \Big\{\varepsilon \in \mathcal{R}(\mathcal{S} \cup \mathcal{S}^0 \cup \mathcal{S}^1 \cup \mathcal{S}^2 \cup \mathcal{S}^3) \mid \varepsilon_{(01)(10)} \neq 0\Big\}$
   $\mathcal{S}^4:\ \varepsilon_{(01)(10)A} = 0 \quad \forall A \in H_3$

Step 1. In the rest of this subsection the parameters $a, b, c, \dots$ are arbitrary complex numbers. Explicit solution under the assumption $\varepsilon_{(01)(10)} \neq 0$, $\varepsilon_{(22)(21)} \neq 0$ can be written as four parametric matrices. These matrices in $\mathcal{R}^0$ can be equivalently replaced by renormalized matrices $\mathcal{R}^0_{nor} = \{\varepsilon^0_1, \varepsilon^0_2, \varepsilon^0_3, \varepsilon^0_4\}$

where

$$\varepsilon_1^0 = \begin{pmatrix} 0&0&1&0&0&1&a&b \\ 0&0&0&0&0&0&c&d \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&1 \\ a&c&0&0&0&0&0&0 \\ b&d&0&1&0&1&0&0 \end{pmatrix}, \qquad \varepsilon_2^0 = \begin{pmatrix} 0&0&1&d&ad&1&1&a \\ 0&0&bad&bad&bad&bd&b&ab \\ 1&bad&0&0&dcba&c&cba&cba \\ d&bad&0&0&dcba&d&c&1 \\ ad&bad&dcba&dcba&&0&cba&ac \\ 1&bd&c&d&0&0&cb&1 \\ 1&b&cba&c&cba&cb&0&0 \\ a&ab&cba&1&ac&1&0&0 \end{pmatrix},$$

$$\varepsilon_3^0 = \begin{pmatrix} 0&0&1&0&0&1&0&0 \\ 0&0&bd&0&0&ad&a&b \\ 1&bd&0&0&0&c&0&cb \\ 0&0&0&0&0&d&0&1 \\ 0&0&0&0&0&0&0&0 \\ 1&ad&c&d&0&0&ac&1 \\ 0&a&0&0&0&ac&0&0 \\ 0&b&cb&1&0&1&0&0 \end{pmatrix}, \quad \varepsilon_4^0 = \begin{pmatrix} 0&0&1&0&ad&1&0&a \\ 0&0&bd&0&0&0&0&b \\ 1&bd&0&0&0&c&0&cb \\ 0&0&0&0&0&d&0&1 \\ ad&0&0&0&0&0&0&ac \\ 1&0&c&d&0&0&0&1 \\ 0&0&0&0&0&0&0&0 \\ a&b&cb&1&ac&1&0&0 \end{pmatrix}.$$

Step 2. Note that the system of 48 equations $\mathcal{S}^0$ together with $\varepsilon_{(01)(10)} \neq 0$ enforces zeros on all positions from $\mathcal{I}_1^k$. Moreover, the assumption $\varepsilon_{(10)(11)} \neq 0$ and $\mathcal{S}^0$ enforces further 4 zeros. Then the assumption $\varepsilon_{(01)(10)} \neq 0$, $\varepsilon_{(10)(11)} \neq 0$, $\varepsilon_{(01)(22)} \neq 0$ gives us a single solution:

$$\mathcal{R}^1_{nor} = \{\varepsilon^1\}, \qquad \text{where} \quad \varepsilon^1 = \begin{pmatrix} 0&0&1&a&b&1&0&0 \\ 0&0&c&0&0&d&0&0 \\ 1&c&0&0&1&e&0&0 \\ a&0&0&0&0&f&0&0 \\ b&0&1&0&0&0&0&0 \\ 1&d&e&f&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

Step 3. Further solutions with assumption $\varepsilon_{(01)(10)} \neq 0$, $\varepsilon_{(10)(11)} \neq 0$, inequivalent to those in $\mathcal{R}^1$ and $\mathcal{R}^0$, are listed below:

$$\mathcal{R}^2_{nor} = \{\varepsilon^2_1, \varepsilon^2_2, \varepsilon^2_3, \varepsilon^2_4, \varepsilon^2_5, \varepsilon^2_6, \varepsilon^2_7, \varepsilon^2_8\}$$

$$\varepsilon_1^2 = \begin{pmatrix} 0&0&1&a&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&1&b&0&c \\ a&0&0&0&ac&d&e&ac \\ 0&0&1&ac&0&0&0&0 \\ 0&0&b&d&0&0&0&0 \\ 0&0&0&e&0&0&0&0 \\ 0&0&c&ac&0&0&0&0 \end{pmatrix}, \quad \varepsilon_2^2 = \begin{pmatrix} 0&0&1&a&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&1&b&1&d \\ a&0&0&0&ad&0&c&ad \\ 0&0&1&ad&0&0&0&0 \\ 0&0&b&0&0&0&0&0 \\ 0&0&1&c&0&0&0&0 \\ 0&0&d&ad&0&0&0&0 \end{pmatrix}, \qquad \varepsilon_3^2 = \begin{pmatrix} 0&0&1&a&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 1&1&0&0&1&b&0&c \\ a&0&0&0&ac&d&0&ac \\ 0&0&1&ac&0&0&0&0 \\ 0&0&b&d&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&c&ac&0&0&0&0 \end{pmatrix},$$

$$\varepsilon_4^2 = \begin{pmatrix} 0&0&1&a&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 1&1&0&0&1&b&1&c \\ a&0&0&0&ac&0&0&ac \\ 0&0&1&ac&0&0&0&0 \\ 0&0&b&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&c&ac&0&0&0&0 \end{pmatrix}, \quad \varepsilon_5^2 = \begin{pmatrix} 0&0&1&a&0&0&0&0 \\ 0&0&1&1&0&0&0&0 \\ 1&1&0&0&1&bd&b&c \\ a&1&0&0&ac&d&bd&ac \\ 0&0&1&ac&0&0&0&0 \\ 0&0&bd&d&0&0&0&0 \\ 0&0&b&bd&0&0&0&0 \\ 0&0&c&ac&0&0&0&0 \end{pmatrix}, \quad \varepsilon_6^2 = \begin{pmatrix} 0&0&1&a&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 1&0&0&0&1&0&0&b \\ a&1&0&0&ab&c&d&ab \\ 0&0&1&ab&0&0&0&0 \\ 0&0&0&c&0&0&0&0 \\ 0&0&0&d&0&0&0&0 \\ 0&0&b&ab&0&0&0&0 \end{pmatrix},$$

$$\varepsilon_7^2 = \begin{pmatrix} 0&0&1&a&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 1&0&0&0&1&0&1&b \\ a&1&0&0&ab&0&c&ab \\ 0&0&1&ab&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&1&c&0&0&0&0 \\ 0&0&b&ab&0&0&0&0 \end{pmatrix}, \quad \varepsilon_8^2 = \begin{pmatrix} 0&0&1&a&b&0&0&0 \\ 0&0&c&d&0&e&0&0 \\ 1&c&0&0&1&0&0&0 \\ a&d&0&0&0&f&0&0 \\ b&0&1&0&0&0&0&0 \\ 0&e&0&f&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}.$$

Step 4. Now we can of course ignore the equations $\mathcal{S}^1$ because they are satisfied identically due to the system $\mathcal{S}^2$. We list the next set

$$\mathcal{R}^3_{nor} = \{\varepsilon^3_1, \varepsilon^3_2, \varepsilon^3_3, \varepsilon^3_4\}$$

$$\varepsilon_1^3 = \begin{pmatrix} 0&0&1&a&0&0&0&0 \\ 0&0&0&0&0&1&1&0 \\ 1&0&0&0&0&0&0&0 \\ a&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \quad \varepsilon_2^3 = \begin{pmatrix} 0&0&1&a&0&0&0&0 \\ 0&0&b&0&0&1&0&0 \\ 1&b&0&0&0&0&0&0 \\ a&0&0&0&0&c&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&1&0&c&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix},$$

$$\varepsilon_3^3 = \begin{pmatrix} 0&0&1&0&0&1&0&0 \\ 0&0&0&0&0&1&1&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \quad \varepsilon_4^3 = \begin{pmatrix} 0&0&1&0&0&1&0&0 \\ 0&0&a&0&0&1&0&0 \\ 1&a&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}.$$

Step 5. The systems $\mathcal{S}^0$, $\mathcal{S}^2$, $\mathcal{S}^3$ together with $\varepsilon_{(01)(10)} \neq 0$ give us 12 zeros and further 20 non-trivial conditions. Adding two zeros following from $\mathcal{S}$ we obtain 3 solutions:

$$\mathcal{R}^4_{nor} = \{\varepsilon_1^4, \varepsilon_2^4, \varepsilon_3^4\}$$

$$\varepsilon_1^4 = \begin{pmatrix} 0&0&1&0&0&0&0&a \\ 0&0&0&b&0&0&c&0 \\ 1&0&0&0&0&0&d&0 \\ 0&b&0&0&0&0&0&e \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&c&d&0&0&0&0&0 \\ a&0&0&e&0&0&0&0 \end{pmatrix}, \quad \varepsilon_2^4 = \begin{pmatrix} 0&0&1&a&0&0&0&0 \\ 0&0&b&c&0&0&0&0 \\ 1&b&0&0&0&0&0&0 \\ a&c&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \quad \varepsilon_3^4 = \begin{pmatrix} 0&0&1&0&0&a&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&b&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ a&0&b&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}.$$

Since all pairs of relevant indices lie in *one* orbit (the whole set $\mathcal{I}$) the system $\mathcal{S}^4 : \varepsilon_k = 0, \forall k \in \mathcal{I}$ enforces zeros on all 24 positions. This precisely means that now *only the trivial zero solution is inequivalent to solutions in* $\mathcal{R}^0$, $\mathcal{R}^1$, $\mathcal{R}^2$, $\mathcal{R}^3$, $\mathcal{R}^4$*, i.e. we have evaluated the whole* $\mathcal{R}(\mathcal{S})$ *up to equivalence.*

Our final goal in this part. is to compute the complete set of inequivalent *normalized* solutions. Thus in the final stage we took each solution matrix and discussed all possible combinations of zero or non-zero parameters like in the following example.

Example 8 *For instance take the matrix* ε_2^0 *and let all its parameters be non-vanishing. Our question is whether or not it is possible to normalize it to the trivial contraction matrix* ε_0 *which has all relevant epsilons equal to unity. Then the resulting graded contractions would be isomorphic to the algebra* $sl(3,\mathbb{C})$ *for arbitrary non-zero values of parameters in* ε_2^0*. We have verified that the system of 24 equations corresponding to the matrix equality* $\varepsilon_2^0 \bullet \alpha = \varepsilon_0$ *has a general solution in* $\mathbb{C}\backslash\{0\}$*. The matrix* ε_2^0 *with non–zero parameters is then equivalent to the trivial solution* ε_0 *and the corresponding graded contraction is isomorphic to* $sl(3,\mathbb{C})$*.*

Similar calculations had to be done for all matrices $\varepsilon_1^0, \ldots, \varepsilon_3^4$. Results are given in the Appendix.

7 Algorithm of identification

The goal of our work is to determine the structure of Lie algebras from the structure constants. Each of the 188 contraction matrices found there uniquely determines a set of structure constants of an 8-dimensional Lie algebra. Unfortunately existing methods ([32,33] and references cited therein) do not allow to recognize all 88 solvable indecomposable algebras found here among the 188 cases. Therefore practical necessity of this paper was to develop further the identification algorithm.

In this section, after recalling pertinent properties of Lie algebras, we describe seven steps of our algorithm.

Let $\mathcal{L}$ be a complex Lie algebra of dimension n. We denote the **derived algebra** of $\mathcal{L}$ by

$$D(\mathcal{L}) = [\mathcal{L}, \mathcal{L}],$$

and the **center** of $\mathcal{L}$ by

$$C(\mathcal{L}) = \{x \in \mathcal{L} \mid \forall y \in \mathcal{L},\ [x, y] = 0\}.$$

If $\mathcal{L}_1, \mathcal{L}_2 \subset \mathcal{L}$ are subalgebras of $\mathcal{L}$, such that $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ (as vector spaces), then the Lie algebra $\mathcal{L}$ is:

(1) the direct sum of its ideals $\mathcal{L}_1, \mathcal{L}_2$, denoted $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$, if

$$[\mathcal{L}_1, \mathcal{L}_2] = 0, \qquad [\mathcal{L}_i, \mathcal{L}_i] \subseteq \mathcal{L}_i, \qquad i = 1, 2,$$

(2) the semidirect sum of its ideal $\mathcal{L}_1$ and subalgebra $\mathcal{L}_2$, denoted $\mathcal{L} = \mathcal{L}_1 \in\!\!\!\!+\, \mathcal{L}_2$, if

$$[\mathcal{L}_1, \mathcal{L}_2] \subseteq \mathcal{L}_1, \qquad [\mathcal{L}_i, \mathcal{L}_i] \subseteq \mathcal{L}_i, \qquad i = 1, 2.$$

We now describe the method of identification in particular steps.

Step 1. Splitting of the maximal central component.

If the complement $X = C(\mathcal{L}) \backslash D(\mathcal{L})$ of the derived algebra to the center is nonempty, then the central decomposition of the Lie algebra $\mathcal{L}$ can be obtained from the decomposition of the quotient algebra

$$\mathcal{L}/D(\mathcal{L}) = X/D(\mathcal{L}) \oplus \widetilde{\mathcal{L}}/D(\mathcal{L}), \tag{46}$$

where $X/D(\mathcal{L})$ denotes the set of all cosets modulo $D(\mathcal{L})$ that are represented by the elements of X and $\widetilde{\mathcal{L}}/D(\mathcal{L})$ is the complement of $X/D(\mathcal{L})$ in $\mathcal{L}/D(\mathcal{L})$, where $\widetilde{\mathcal{L}}$ contains $D(\mathcal{L})$. From now on we consider Lie algebras without a separable central component. For those algebras $C(\mathcal{L}) \subseteq D(\mathcal{L})$ holds.

Step 2. Decomposition into a direct sum of indecomposable ideals.

Let

$$C_R(\mathrm{ad}(\mathcal{L})) = \{x \in R \mid \forall y \in \mathrm{ad}(\mathcal{L}),\ [x, y] = 0\} \tag{47}$$

denote the centralizer of the adjoint representation of $\mathcal{L}$ in the ring $R = \mathbb{C}^{n,n}$. An idempotent $E \in R$ in the ring R is a nonzero matrix satisfying $E^2 = E$. Lie algebra $\mathcal{L}$ is decomposable into a direct sum of its ideals if and only if there exists a non–trivial idempotent $1 \neq E \in C_R(\mathrm{ad}(\mathcal{L}))$. In such a case the decomposition has the form

$$\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1, \qquad [\mathcal{L}_0, \mathcal{L}_1] = 0, \qquad [\mathcal{L}_i, \mathcal{L}_i] \subseteq \mathcal{L}_i, \qquad i = 0, 1,$$

where $\mathcal{L}_0, \mathcal{L}_1$ are eigen-subspaces of the idempotent E corresponding to the eigenvalues 0,1. In this step, all decomposable algebras are decomposed and from now on we consider indecomposable algebras only. For an explicit algorithm of finding a nontrivial idempotent see [32].

Step 3. Determination of the radical, the Levi decomposition and the nilradical.

First we find the radicals for all algebras. The **radical** $R(\mathcal{L})$ of the Lie algebra $\mathcal{L}$ is the maximal solvable ideal in $\mathcal{L}$ and in our case

$$R(\mathcal{L}) = \{x \in \mathcal{L} \mid \forall y \in D(\mathcal{L}),\ \mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) = 0\}. \tag{48}$$

Then we use Levi's theorem which states that for an arbitrary finite dimensional Lie algebra $\mathcal{L}$ over a field of characteristic zero there exists a semisimple subalgebra $\mathcal{S}$ such that

$$\mathcal{L} = R(\mathcal{L}) \ni\!\!\!\!+ \mathcal{S}, \tag{49}$$

and we make the semidirect decomposition of all Lie algebras. If $R(\mathcal{L}) = 0$, the Lie algebra $\mathcal{L}$ is semisimple and if $\mathcal{S} = 0$, it is solvable. At last we find the maximal nilpotent ideals (**nilradicals**) from radicals of all algebras and determine which algebras are nilpotent. The algorithm for providing Levi's decompositions and algorithm of computing the nilradical are completely described in [32].

Step 4. Computation of the derived series, the lower central series and the upper central series.

The **derived series** of Lie algebra $\mathcal{L}$ is a series of ideals $D^0(\mathcal{L}) \supseteq D^1(\mathcal{L}) \supseteq D^2(\mathcal{L}) \supseteq \ldots \supseteq D^k(\mathcal{L}) \supseteq \ldots$ defined by:

$$D^0(\mathcal{L}) = \mathcal{L}, \qquad D^{k+1}(\mathcal{L}) = [D^k(\mathcal{L}), D^k(\mathcal{L})], \qquad k = 0, 1, 2, \ldots.$$

The **lower central series** of $\mathcal{L}$ is a series of ideals $\mathcal{L}^0 \supseteq \mathcal{L}^1 \supseteq \mathcal{L}^2 \supseteq \ldots \supseteq$

$\mathcal{L}^k \supseteq \ldots$ defined by:

$$\mathcal{L}^0 = \mathcal{L}, \qquad \mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}], \qquad k = 0, 1, 2, \ldots.$$

The **upper central series** of $\mathcal{L}$ is a series of ideals $C^0(\mathcal{L}) \subseteq C^1(\mathcal{L}) \subseteq C^2(\mathcal{L}) \subseteq \ldots \subseteq C^k(\mathcal{L}) \subseteq \ldots$ defined by:

$$C^0(\mathcal{L}) = 0, \qquad C^{k+1}(\mathcal{L})/C^k(\mathcal{L}) = C(\mathcal{L}/C^k(\mathcal{L})), \qquad k = 0, 1, 2, \ldots.$$

The dimensions of ideals in the above series are invariants of $\mathcal{L}$. In Step 4 we divide all Lie algebras into classes according to these invariants.

Step 5. Determination of the algebra of derivations.

A linear mapping $d : \mathcal{L} \to \mathcal{L}$ is called a **derivation** of the Lie algebra $\mathcal{L}$ if

$$d[x, y] = [dx, y] + [x, dy], \qquad \forall x, y \in \mathcal{L}. \tag{50}$$

The set of all derivations of $\mathcal{L}$ forms a Lie algebra $\mathrm{Der}(\mathcal{L}) \subseteq gl(\mathcal{L})$ called **Lie algebra of derivations** of $\mathcal{L}$. If $c_{i,j}^k$ are the structure constants of $\mathcal{L}$ in the basis $\{e_i\}_{i=1}^n$ and let $de_i = \sum_{j=1}^n d_{ji} e_j$, $\forall i \in \hat{n} = \{1, 2, \ldots, n\}$, then $d \in \mathrm{Der}(\mathcal{L})$, if

$$\sum_{m=1}^n (c_{ij}^m d_{km} - c_{mj}^k d_{mi} - c_{im}^k d_{mj}) = 0, \qquad \forall i, j, k \in \hat{n}. \tag{51}$$

The dimension of the algebra of derivations is an invariant for the Lie algebra $\mathcal{L}$, and we divide each class from the previous step into new classes according to $\dim(\mathrm{Der}(\mathcal{L}))$.

Step 6. Casimir operators.

An element F of the universal enveloping algebra of $\mathcal{L}$ which satisfies

$$[x, F] = 0, \qquad \forall x \in \mathcal{L}, \tag{52}$$

is called Casimir operator [34,35]. These operators can be calculated as follows. Represent the elements of basis $\{e_i\}_{i=1}^n$ of $\mathcal{L}$ by the vector fields

$$e_i \to \hat{x}_i = \sum_{j,k=1}^n c_{ij}^k x_k \frac{\partial}{\partial x_j}, \tag{53}$$

which have the same commutation rules and act on the space of continuously differentiable functions $F(x_1, \ldots, x_n)$. A function F is called **formal invariant** of $\mathcal{L}$ if it is a solution of

$$\hat{x}_i F = 0, \qquad i \in \hat{n}. \tag{54}$$

The number of algebraically independent formal invariants is

$$\tau(\mathcal{L}) = \dim(\mathcal{L}) - r(\mathcal{L}), \tag{55}$$

where $r(\mathcal{L})$ is the rank of the antisymmetric matrix $M_{\mathcal{L}}$ with entries $(M_{\mathcal{L}})_{ij} = \sum_k c_{ij}^k e_k$:

$$r(\mathcal{L}) = \sup_{(e_1,\ldots,e_n)} \operatorname{rank}(M_{\mathcal{L}}). \tag{56}$$

From a polynomial formal invariant $F(x_1, \ldots, x_n)$ the Casimir invariant is obtained according to the recipe: replace the variables x_i by noncommuting elements e_i and symmetrize $F(e_1, \ldots, e_n)$ [34]. In the case of nilpotent algebras all solutions $F(x_1, \ldots, x_n)$ can be written as polynomial functions yielding Casimir operators. We can decide whether two algebras are non-isomorphic by comparing the numbers of their formal invariants or, in the case of polynomial invariants, according to their order.

Step 7. Seeking isomorphisms in the classes of algebras.

Complex Lie algebras $\mathcal{L}$ and $\mathcal{L}'$ of the same dimension n determined by structure constants x_{ij}^k and y_{ij}^k are isomorphic if there exists a regular matrix $A \in \mathbb{C}^{n,n}$, whose elements satisfy the system of $n^2(n-1)/2$ quadratic equations

$$\sum_{r=1}^{n} x_{ij}^r A_{kr} = \sum_{\mu,\nu=1}^{n} A_{\mu i} A_{\nu j} y_{\mu\nu}^k, \qquad i = 1, \ldots, n-1, \quad j = i, \ldots, n, \quad k \in \hat{n}. \tag{57}$$

We can either solve this system on computer and test solutions on regularity or try to prove that this system has no regular solution.

8 Results of the identification

The set of 188 inequivalent solutions of the system of contraction equations for the Pauli graded Lie algebra $sl(3, \mathbb{C})$ was divided into 13 groups according to the numbers ν of zeros among 24 relevant entries in the contraction matrices ε. These solutions are denoted $\varepsilon^{\nu,i}$, where the second index i is numbering solutions with the same ν. Correspondingly, the contracted Lie algebra corresponding to solution $\varepsilon^{\nu,i}$ is denoted $\mathcal{L}_{\nu,i}$. The following table gives the numbers of solutions ε corresponding to each ν:

ν	0	9	12	15	16	17	18	19	20	21	22	23	24
$1 \le i \le$	1	1	2	7	7	17	36	45	42	21	7	1	1

Among the 188 solutions there are two trivial solutions. One trivial solution $\varepsilon^{24,1}$, with 24 zeros, corresponds to the 8–dimensional Abelian Lie algebra while the other trivial solution $\varepsilon^{0,1}$, without zeros, corresponds to the initial

Lie algebra $sl(3, \mathbb{C})$. From the remaining 186 nontrivial solutions, 11 solutions depend on one non-zero complex parameter a and two depend on two non-zero complex parameters a, b. The corresponding parametric algebras are denoted by $\mathcal{L}_{\nu,i}(a)$, $\mathcal{L}_{\nu,i}(a, b)$.

In Step 1, 71 algebras allowed the separation of a central component. For these algebras only non-Abelian parts with dimension lower than 8 were further investigated. We will denote the non-Abelian part of the Lie algebra $\mathcal{L}$ by $\mathcal{L}'$.

In Step 2 only 12 algebras were decomposable. Decomposition was in all cases the direct sum of two indecomposable ideals. At this stage we are left with 198 indecomposable algebras for the next steps leading to their identification.

The Levi decomposition in Step 3 was trivial for all algebras in the sense that there is no semisimple part. Of the resulting Lie algebras 24 were solvable (non-nilpotent) and 174 nilpotent.

In Step 4, computation of dimensions of the derived series, the lower central series and the upper central series improved the accuracy of determination of the number of Lie algebras: solvable algebras have split into 18 classes and nilpotent algebras into 52 classes, to be still refined in the following steps.

Dimensions of algebras of derivations in Step 5 divided nilpotent algebras into 100 classes and retained 18 classes of solvable algebras.

In Step 6 the numbers of formal invariants of Lie algebras and of Casimir invariants for the nilpotent Lie algebras were computed. It lead to 115 classes of nilpotent algebras. The Casimir invariants were computed straightforwardly from the definition (52).

In Step 7, computer computation discovered 4 isomorphic pairs among the solvable algebras and 52 isomorphic pairs among the nilpotent algebras. Furthermore, among 11 parametric algebras, two solvable algebras and two nilpotent ones were extended to zero values of parameters:

$$\begin{aligned} &\mathcal{L}_{15,6}(0) \cong \mathcal{L}_{16,6}, \qquad \mathcal{L}_{15,5}(a,0) \cong \mathcal{L}_{16,2}(\tfrac{1}{a}), \\ &\mathcal{L}_{12,2}(0,b) \cong \mathcal{L}_{15,7}(\tfrac{1}{b^2}),\ \mathcal{L}_{18,25}(0) \cong \mathcal{L}_{19,26}, \end{aligned} \tag{58}$$

So we obtained 18 non-isomorphic (non-nilpotent) solvable algebras. Moreover the structure of all decomposable algebras was completed. We give the

structure of mutually non-isomorphic decomposable algebras:

$$\begin{aligned}
&\mathcal{L}_{18,32} = \mathcal{L}'_{21,9} \oplus \mathcal{L}'_{21,9},\ \mathcal{L}_{19,36} = \mathcal{L}'_{21,9} \oplus \mathcal{L}'_{22,1},\ \mathcal{L}_{20,10} = \mathcal{L}'_{23,1} \oplus \mathcal{L}'_{21,2},\\
&\mathcal{L}'_{20,20} = \mathcal{L}'_{23,1} \oplus \mathcal{L}'_{21,9},\ \mathcal{L}_{20,21} = \mathcal{L}'_{22,1} \oplus \mathcal{L}'_{22,1},\ \mathcal{L}'_{21,4} = \mathcal{L}'_{23,1} \oplus \mathcal{L}'_{22,1},\\
&\mathcal{L}_{21,15} = \mathcal{L}'_{23,1} \oplus \mathcal{L}'_{22,3},\ \mathcal{L}'_{22,2} = \mathcal{L}'_{23,1} \oplus \mathcal{L}'_{23,1},
\end{aligned} \tag{59}$$

as well as the list of all isomorphisms among non-decomposed algebras:

$$\begin{array}{lll}
\mathcal{L}'_{22,2} \cong \mathcal{L}'_{22,6}, & \mathcal{L}_{20,21} \cong \mathcal{L}_{20,22}, & \mathcal{L}'_{21,4} \cong \mathcal{L}'_{21,6} \cong \mathcal{L}'_{21,10},\\
\mathcal{L}'_{22,3} \cong \mathcal{L}'_{22,4} \cong \mathcal{L}'_{22,5}, & \mathcal{L}'_{21,3} \cong \mathcal{L}'_{21,5} \cong \mathcal{L}'_{21,8}, & \mathcal{L}'_{21,11} \cong \mathcal{L}'_{21,13} \cong \mathcal{L}'_{21,18},\\
\mathcal{L}'_{21,12} \cong \mathcal{L}'_{21,14} \cong \mathcal{L}'_{21,17}, & \mathcal{L}'_{20,14} \cong \mathcal{L}'_{20,23} \cong \mathcal{L}'_{20,28}, & \mathcal{L}'_{20,19} \cong \mathcal{L}'_{20,26} \cong \mathcal{L}'_{20,29},\\
\mathcal{L}'_{20,13} \cong \mathcal{L}'_{20,15} \cong \mathcal{L}'_{20,24}, & \mathcal{L}_{20,32} \cong \mathcal{L}_{20,37}, & \mathcal{L}_{20,31} \cong \mathcal{L}_{20,34},\\
\mathcal{L}_{19,31} \cong \mathcal{L}_{19,33} \cong \mathcal{L}_{19,39}, & \mathcal{L}_{20,17} \cong \mathcal{L}_{20,18} \cong \mathcal{L}_{20,27}, & \mathcal{L}_{20,33} \cong \mathcal{L}_{20,36},\\
\mathcal{L}_{19,25} \cong \mathcal{L}_{19,38} \cong \mathcal{L}_{19,40}, & \mathcal{L}_{20,12} \cong \mathcal{L}_{20,25} \cong \mathcal{L}_{20,30}, & \mathcal{L}_{18,26} \cong \mathcal{L}_{18,31},\\
\mathcal{L}_{19,28} \cong \mathcal{L}_{19,34} \cong \mathcal{L}_{19,35}, & \mathcal{L}_{18,25}(a) \cong \mathcal{L}_{18,30}(a) \cong \mathcal{L}_{18,33}(a), & \mathcal{L}_{19,26} \cong \mathcal{L}_{19,30} \cong \mathcal{L}_{19,37}.
\end{array}$$

9 Algebra of derivations

Computation of the set of invariants according to [32] turned out to be insufficient for our purpose: in many cases there were several Lie algebras with the same characteristics proposed in [32]. Even if Casimir operators were used, we did not attain unique characterization. Unexpectedly, additional computation of the dimensions of algebras of derivations finally almost solved the problem.

There remained still the following undetermined cases among the nilpotent algebras:

$$\begin{gathered}
\mathcal{L}_{17,2} \text{ and } \mathcal{L}_{19,22}, \qquad \mathcal{L}_{17,13}(a) \text{ and } \mathcal{L}_{18,11}, \qquad \mathcal{L}_{18,16} \text{ and } \mathcal{L}_{19,20},\\
\\
\mathcal{L}_{15,5}(a,b) \text{ and } \mathcal{L}_{17,10}, \qquad \mathcal{L}_{16,3}(a) \text{ and } \mathcal{L}_{17,9} \text{ and } \mathcal{L}_{17,12}.
\end{gathered} \tag{60}$$

Determination of the algebra of derivations is a linear problem. If two Lie algebras are isomorphic, then their algebras of derivations must be isomorphic too. So we can apply our algorithm of identification to algebras of derivations of Lie algebras (60). Obtained characterizations of algebras of derivations are given in Table 3. All investigated algebras of derivations were indecomposable and without separable central component. The dimensions of their derived

series, lower central series and upper central series are given in the 2nd, 3rd and 4th column. The numbers τ of formal invariants of algebras of derivations are given in the 5th column. We constructed also the **sequences of algebras of derivations**

$$\mathrm{Der}^k(\mathcal{L}) = \underbrace{\mathrm{Der}(\ldots(\mathrm{Der}(\mathcal{L})))}_{k-times}, \qquad k \in \mathbb{N}. \tag{61}$$

Dimensions of members of these sequences are in the 6th column of the Table 3. In the same way like for the series in the 2nd, 3rd and 4th column, we did not write out the repeating numbers at the end of sequence.

$\mathcal{L}$	$D^k(\mathrm{Der}(\mathcal{L}))$	$(\mathrm{Der}(\mathcal{L}))^k$	$C^k(\mathrm{Der}(\mathcal{L}))$	τ	$\mathrm{Der}^k(\mathcal{L})$
$\mathcal{L}_{17,2}$	(17,15)	(17,15)	(0)	3	17,19
$\mathcal{L}_{19,22}$	(17,14,8,0)	(17,14)	(0)	3	17,19
$\mathcal{L}_{15,5}(a,b),\ {}^{a\neq 0}_{(-1,0)\neq(a,b)\neq(1,1)}$	(16,15,6,0)	(16,15)	(0)	6	16
$\mathcal{L}_{17,10}$	(16,15,6,0)	(16,15)	(0)	6	16,19
$\mathcal{L}_{17,13}(a),\ 0 \neq a \neq -1$	(17,15,6,0)	(17,15)	(0)	5	17
$\mathcal{L}_{18,11}$	(17,15,6,0)	(17,15)	(0)	5	17,18
$\mathcal{L}_{18,16}$	(19,17,12,3,0)	(19,17)	(0)	7	19
$\mathcal{L}_{19,20}$	(19,16,9,0)	(19,16)	(0)	7	19,25
$\mathcal{L}_{16,3}(a),\ a \neq 0$	(16,15,6,0)	(16,15)	(0)	6	16
$\mathcal{L}_{17,9}$	(16,15,6,0)	(16,15)	(0)	6	16
$\mathcal{L}_{17,12}$	(16,15,6,0)	(16,15)	(0)	6	16

Table 3
Characterizations of algebras of derivations: derived series, lower central series, upper central series, numbers of formal invariants and sequences of algebras of derivations.

Table 3 shows that there remains the unresolved case of three nilpotent Lie algebras $\mathcal{L}_{16,3}(a)$, $\mathcal{L}_{17,9}$ and $\mathcal{L}_{17,12}$; other algebras are non-isomorphic. The

nonzero commutation relations of these Lie algebras are:

$$\begin{aligned}
&\mathcal{L}_{16,3}(a) && [e_1,e_3]=-ae_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,\ [e_1,e_6]=e_4,\\
& && [e_2,e_3]=e_7,\ [e_2,e_6]=e_8,\ [e_3,e_5]=e_8,\ [e_4,e_6]=e_7\\[2ex]
&\mathcal{L}_{17,9} && [e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,\ [e_1,e_6]=e_4,\\
& && [e_2,e_3]=e_7,\ [e_3,e_5]=e_8,\ [e_4,e_6]=e_7\\[2ex]
&\mathcal{L}_{17,12} && [e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_6]=e_4,\ [e_2,e_3]=e_7,\\
& && [e_2,e_6]=e_8,\ [e_3,e_5]=e_8,\ [e_4,e_6]=e_7
\end{aligned} \tag{62}$$

Let us first prove that Lie algebras $\mathcal{L}_{17,9}$ and $\mathcal{L}_{17,12}$ are not isomorphic. The system (57) has in this case 140 equations. Some equations are of the form $A_{i,j}=0$. After multiple use of these equations in the system there remain only 40 equations and the matrix of isomorphism $A \in \mathbb{C}^{8,8}$ has the form

$$\begin{pmatrix}
A_{1,1} & A_{1,2} & A_{1,3} & 0 & 0 & A_{1,6} & 0 & 0\\
A_{2,1} & A_{2,2} & A_{2,3} & 0 & 0 & A_{2,6} & 0 & 0\\
A_{3,1} & A_{3,2} & A_{3,3} & 0 & 0 & A_{3,6} & 0 & 0\\
A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} & A_{4,5} & A_{4,6} & 0 & 0\\
A_{5,1} & A_{5,2} & A_{5,3} & A_{5,4} & A_{5,5} & A_{5,6} & 0 & 0\\
A_{6,1} & A_{6,2} & A_{6,3} & 0 & 0 & A_{6,6} & 0 & 0\\
A_{7,1} & A_{7,2} & A_{7,3} & A_{7,4} & A_{7,5} & A_{7,6} & A_{7,7} & A_{7,8}\\
A_{8,1} & A_{8,2} & A_{8,3} & A_{8,4} & A_{8,5} & A_{8,6} & A_{8,7} & A_{8,8}
\end{pmatrix}. \tag{63}$$

The simplest equations of the system are

$$\begin{gathered}
A_{6,2}A_{4,4}=0, \quad A_{6,2}A_{4,5}=0, \quad A_{6,3}A_{4,4}=0, \quad A_{6,6}A_{4,5}=0,\\[2ex]
A_{6,1}A_{4,4}=A_{7,8}, \quad A_{6,1}A_{4,5}=A_{7,7}, \quad A_{6,3}A_{4,5}=A_{7,8}, \quad A_{6,6}A_{4,4}=A_{7,7}.
\end{gathered} \tag{64}$$

If we put $A_{4,4}=0$ then $A_{7,7}=0,\ A_{7,8}=0$ and the matrix A is singular. On the other hand $A_{4,4}\neq 0$ implies $A_{6,3}=0$ and consequently $A_{7,8}=0$. From $A_{4,4}\neq 0$ and $A_{7,8}=0$ now follows $A_{6,1}=0$ and consequently $A_{7,7}=0$. Thus the matrix A is again singular and the proof is completed.

In the remaining cases $\mathcal{L}_{16,3}(a)$ *vs.* $\mathcal{L}_{17,9}$ and $\mathcal{L}_{16,3}(a)$ *vs.* $\mathcal{L}_{17,12}$ the matrices of isomorphism have the same form as (63). Moreover in the case $\mathcal{L}_{16,3}(a)$ *vs.* $\mathcal{L}_{17,12}$ there are the equations (64) in system (57) and therefore the Lie algebras $\mathcal{L}_{16,3}(a)$ and $\mathcal{L}_{17,12}$ are non-isomorphic. The proof that the Lie algebras $\mathcal{L}_{16,3}(a)$ and $\mathcal{L}_{17,9}$ are non-isomorphic is similar to the first one, but more complicated and we shall not describe it there. This finished the identification.

Table 4 summarizes our results. For each dimension of non-Abelian part of Lie algebra in the first column, the numbers of obtained Lie algebras are given in the other columns according to their types. Together with the 8-

Dimension of non–Abelian part	Solvable		Nilpotent		Total
	Indecomp.	Decomp.	Indecomp.	Decomp.	
3			1		1
4	1		1		2
5	1		4		5
6	1		9	1	11
7	4	1	28	1	34
8	11	2	77	3	93

Table 4
The numbers of contracted Lie algebras from Pauli graded $sl(3,\mathbb{C})$*.*

dimensional Abelian Lie algebra and the Lie algebra $sl(3,\mathbb{C})$ the final number of all contracted Lie algebras is 148.

10 Concluding remarks

- Using the symmetry group of the Pauli grading of $sl(3,\mathbb{C})$, we have evaluated the set of all 188 solutions of the corresponding contraction system up to equivalence. For the solution of the normalization equations and for the explicit evaluation of orbits of solutions we used the computer program MAPLE 8. A new method of solving is based on Theorem 7. It enabled us to check all solutions in the sets $\mathcal{R}^0$, $\mathcal{R}^1$, $\mathcal{R}^2$, $\mathcal{R}^3$ and $\mathcal{R}^4$ also by hands. It is interesting to note that

$$\min\Big\{\nu(\varepsilon)\in\{1,2,3,\dots\}\ \Big|\ \varepsilon\in\mathcal{R}(\mathcal{S})\Big\}=9$$

  i.e. there are no solutions with less than 9 zeros (besides the trivial solution ε_0). Moreover, there are no solutions with 10, 11, 13 or 14 zeros. The complete list of solutions is put in the Appendix. It serves as an input for further analysis — the identification of resulting Lie algebras.
- Method used for identification of resulting Lie algebras was described in Section 7 and 9. For all steps of this method a program in MAPLE 8 was written and applied, in the given order, to each contraction matrix. The number of all non-isomorphic contracted algebras is 148.
- In physical applications, only contractions of the continuous kind are often considered. From the point of view of our method it is only a part of the solution of the contraction system. We have studied this question and the

continuous or discrete type is distinguished by the corresponding subscript of each contraction matrix given in the Appendix.

- Our experience shows that a simple minded application of even powerful symbolic languages does not provide all solutions. The second crucial point is the introduction of equivalence classes of solutions which are difficult to enforce in the symbolic calculation.
- Our results can be compared with graded contractions for the root decomposition [25]. Their investigations resulted in 32 Lie algebras, of which 9 are 8-dimensional non-decomposable.
- Apparently the possibility of three-term contraction equations has not been noticed in the literature before. They may appear for finest gradings, i.e. when the grading subspaces are one-dimensional. Our case of $sl(3,\mathbb{C})$ is too low and all contraction equations reduce to two-term ones. Following example shows that three–term contraction equations may arise for the Pauli grading of $sl(5,\mathbb{C})$. Let us write the contraction equation $e((01)(10)(31))$. Since the subspaces are one-dimensional, we have

$$[X_{01},[X_{10},X_{31}]_\varepsilon]_\varepsilon + [X_{31},[X_{01},X_{10}]_\varepsilon]_\varepsilon + [X_{10},[X_{31},X_{01}]_\varepsilon]_\varepsilon = 0.$$

  Using (4), we obtain a three-term equation

$$\begin{aligned} &\varepsilon_{(01)(10)}\varepsilon_{(11)(31)}(\omega_5-1)(\omega_5^3-\omega_5)+\varepsilon_{(10)(31)}\varepsilon_{(41)(01)}(1-\omega_5)(1-\omega_5^4) \\ +&\varepsilon_{(31)(01)}\varepsilon_{(32)(10)}(1-\omega_5^3)(\omega_5^2-1)=0, \end{aligned}$$

  where ω_5 is the fifth root of unity.
- The ranges of continuous parameters in the 13 contraction matrices where they appear, still need to be restricted by the requirement that for every two values of a parameter one has non-isomorphic Lie algebras.
- The resulting number of non-isomorphic parametric Lie algebras given by one- or two-parameter solutions is 11. They would deserve further study in order to determine the ranges of the parameters. Allowing for zero values of complex parameters a, b we obtained the following isomorphisms:

$$\begin{array}{lll} \mathcal{L}_{12,2}(0,0)=\mathcal{L}_{18,28}, & \mathcal{L}_{12,2}(a,0)\cong\mathcal{L}_{15,6}(a^2), & \mathcal{L}_{12,2}(0,b)\cong\mathcal{L}_{15,7}(\frac{1}{b^2}), \\ \mathcal{L}_{15,5}(0,0)\cong\mathcal{L}_{17,9}, & \mathcal{L}_{15,5}(a,0)\cong\mathcal{L}_{16,2}(\frac{1}{a}), & \mathcal{L}_{15,5}(0,b)\cong\mathcal{L}_{16,3}(b), \\ \mathcal{L}_{15,6}(0)=\mathcal{L}_{16,6}, & \mathcal{L}_{15,7}(0)\cong\mathcal{L}_{16,6}, & \mathcal{L}_{16,1}(0)=\mathcal{L}_{17,3}, \\ \mathcal{L}_{16,2}(0)\cong\mathcal{L}_{17,12}, & \mathcal{L}_{16,3}(0)\cong\mathcal{L}_{17,14}, & \mathcal{L}_{17,7}(0)\cong\mathcal{L}_{18,2}, \\ \mathcal{L}_{17,13}(0)\cong\mathcal{L}_{18,12}, & \mathcal{L}_{18,29}(0)\cong\mathcal{L}_{19,29}, & \mathcal{L}_{18,25}(0)=\mathcal{L}_{19,26}. \end{array}$$

  Here, as elsewhere, $\cong$ denotes isomorphism and $=$ identity of Lie algebras, i.e. the commutation relations are the same. We made the extensions to zero values of parameters, whenever the derived series, the lower central series and the upper central series were the same for nonzero and zero values of parameters (58). However, we were not able to determine exact ranges

of parameters except for the cases of

$\mathcal{L}_{16,3}(a), \mathcal{L}_{17,13}(a), \mathcal{L}_{18,25}(a), \mathcal{L}_{18,29}(a)$, where the range of complex parameter $a \neq 0$ could be restricted to $0 < |a| \leq 1$.

- Let us compare our results with results of [25], where the contractions of $sl(3,\mathbb{C})$ graded by the maximal torus were studied. The toroidal and Pauli gradings of $sl(3,\mathbb{C})$ share one common coarser $\mathbb{Z}_3$–grading (see Fig.1). In both cases the same Lie algebras arise. More precisely, there are the following correspondences (C_k and $A_{m,n}$ are symbols for Lie algebras used in [25] and [35], respectively):

$$
\begin{aligned}
&C_1 \cong \mathcal{L}_{0,1} \cong sl(3,\mathbb{C}), && C_{10} \cong \mathcal{L}_{15,5}(1,1),\\
&C_3 \cong \mathcal{L}_{12,2}(1,1), && C_{18} \cong \mathcal{L}_{18,34} \cong A_{5,33}(a,b) \oplus A_1,\\
&C_7 \cong \mathcal{L}_{9,1}, && C_{28} \cong \mathcal{L}_{21,16} \cong A_{6,3} \oplus 2A_1,\\
&C_8 \cong \mathcal{L}_{18,35}, && C_{32} \cong \mathcal{L}_{24,1} \cong 8A_1.
\end{aligned}
$$

- Moreover there are still other isomorphisms between the results of [25] and the present ones besides the trivial contractions:

$$
\begin{aligned}
&C_9 \cong \mathcal{L}_{20,33}, && C_{30} \cong \mathcal{L}_{22,2} \cong A_{3,1} \oplus A_{3,1} \oplus 2A_1,\\
&C_{27} \cong \mathcal{L}_{21,12}, && C_{31} \cong \mathcal{L}_{23,3} \cong A_{3,1} \oplus 5A_1,\\
&C_{29} \cong \mathcal{L}_{22,3} \cong A_{5,1} \oplus 3A_1.
\end{aligned}
$$

The number of all contracted Lie algebras in [25] is 34. Of these Lie algebras 13 algebras listed above were reobtained in our work.

- It also remains to investigate graded contractions of $sl(3,\mathbb{C})$ starting from the fine gradings which are neither toroidal nor Pauli [23]. Ultimate goal should be a comprehensive description of all graded contractions of $sl(3,\mathbb{C})$. For convenience of the reader we reproduce the two fine gradings of $sl(3,\mathbb{C})$ which are neither Pauli nor toroidal. They are shown in Fig. 1, as Γ_a and Γ_d. For more details see [23]. The former grading group is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$,

$$sl(3,\mathbb{C}) = L_{001} \oplus L_{111} \oplus L_{101} \oplus L_{011} \oplus L_{110} \oplus L_{010} \oplus L_{100},$$

the grading subspaces are generated by the matrices

$$\begin{pmatrix} a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & -a-b \end{pmatrix}, \begin{pmatrix} 0&1&0\\1&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&0&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&1\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&-1&0\\1&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&-1\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&-1\\0&0&0\\1&0&0 \end{pmatrix}.$$

For the latter $\mathbb{Z}_8$-grading we have

$$sl(3,C) = L_0 \oplus L_1 \oplus L_2 \oplus L_3 \oplus L_4 \oplus L_5 \oplus L_6 \oplus L_7,$$

$$\begin{pmatrix} 0&0&0\\0&1&0\\0&0&-1 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&0\\-1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\1&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 2&0&0\\0&-1&0\\0&0&-1 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\-1&0&0\\0&0&0 \end{pmatrix}.$$

- Present work raises further interesting questions.
  - Why, among the contracted Lie algebras, no algebra with non-trivial Lévy decomposition appears?
  - During the contraction the original symmetry group of the grading in general enlarges. One can see it by comparing initial grading symmetry and symmetry group of a contracted Lie algebra containing Abelian algebras.
  - One can study graded contractions preserving a favourite subalgebra with the rest of the Lie algebra fine graded.
  - Simultaneous grading of Lie algebras and their representations allows one to study simultaneous contractions of both: the Lie algebra and its action on the representation space. Thus the representation remains representation of the contracted Lie algebra. It was shown in [31] that the solutions found in this paper can serve also for describing the action of the contracted Lie algebra on its representation space.
  - It would be very interesting to know graded contractions of real forms of $sl(3,\mathbb{C})$ for applications in physics and elsewhere. It is conceivable that they could be studied by splitting each complex solution into several which are valid for one or another real form.

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Appendix A

The complete list of non-equivalent solutions of $\mathcal{S}$ has 188 entries. The list is divided according to the numbers of zeros ν among 24 relevant parameters in the contraction matrices $\varepsilon^{\nu,i}$. Subscript C or D denotes continuous or discrete solution, respectively. In the list $a \neq 0$ and $b \neq 0$ are otherwise arbitrary complex parameters; zeros are shown as dots.

Trivial solutions $\varepsilon^{0,1}$ and $\varepsilon^{24,1}$

$$\begin{pmatrix} \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & 1 \\ 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdot & \cdot & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdot & \cdot & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Solutions with 9 zeros $\varepsilon^{9,1}$

$$\begin{pmatrix} \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \end{pmatrix}_C$$

Solutions with 12 zeros $\varepsilon^{12,1}, \varepsilon^{12,2}$

$$\begin{pmatrix} \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & a & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & b & b & 1 \\ a & 1 & \cdot & \cdot & a & 1 & b & a \\ \cdot & \cdot & 1 & a & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & b & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & a & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_*$$

* $\operatorname{Re} b > 0 \vee (\operatorname{Re} b = 0 \wedge \operatorname{Im} b > 0)$; continuous for $a = 1, b = 1$, otherwise discrete.

Solutions with 15 zeros $\varepsilon^{15,1}, \ldots, \varepsilon^{15,7}$

$$\begin{pmatrix} \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & \cdot & \cdot & 1 & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \end{pmatrix}_C$$

$$\begin{pmatrix} \cdot & \cdot & a & 1 & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ a & 1 & \cdot & \cdot & b & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_{**} \begin{pmatrix} \cdot & \cdot & 1 & a & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot \\ a & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ a & 1 & \cdot & \cdot & 1 & 1 & 1 & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D$$

** Continuous for $a = 1, b = 1$, otherwise discrete.

Solutions with 16 zeros $\varepsilon^{16,1},\ldots,\varepsilon^{16,7}$

$$
\begin{pmatrix}
\cdot&\cdot&1&\cdot&\cdot&1&1&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&a\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&a&\cdot&1&\cdot&1&\cdot&\cdot
\end{pmatrix}_{\dagger}
\begin{pmatrix}
\cdot&\cdot&a&1&1&1&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\
a&1&\cdot&\cdot&1&1&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&1&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot&\cdot&a&1&1&1&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\
a&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&1&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot&\cdot&1&1&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&1&\cdot&1&\cdot&\cdot\\
1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&1&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}_D
$$

$$
\begin{pmatrix}
\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\
1&1&\cdot&\cdot&1&\cdot&\cdot&1\\
1&1&\cdot&\cdot&1&\cdot&\cdot&1\\
\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\
1&1&\cdot&\cdot&1&1&1&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&1&1&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&1&\cdot&\cdot&1&\cdot&\cdot&1\\
1&\cdot&\cdot&\cdot&1&1&\cdot&1\\
\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}_D
$$

† Continuous for $a = 1$, otherwise discrete.

Solutions with 17 zeros $\varepsilon^{17,1},\ldots,\varepsilon^{17,17}$

$$
\begin{pmatrix}
\cdot&\cdot&1&1&\cdot&1&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&1&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&1&\cdot&\cdot&\cdot&1\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot
\end{pmatrix}_C
\begin{pmatrix}
\cdot&\cdot&1&\cdot&\cdot&1&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&1&\cdot&\cdot&\cdot&\cdot&1\\
1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot
\end{pmatrix}_C
\begin{pmatrix}
\cdot&\cdot&1&\cdot&\cdot&1&1&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&1&\cdot&1&\cdot&\cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot&\cdot&1&\cdot&\cdot&1&1&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&1&\cdot&1&\cdot&1&\cdot&\cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot&\cdot&1&\cdot&\cdot&1&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&1&\cdot&1&\cdot&\cdot
\end{pmatrix}_D
$$

$$
\begin{pmatrix}
\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&1&\cdot&1&\cdot&\cdot&\cdot&1\\
\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot
\end{pmatrix}_C
\begin{pmatrix}
\cdot&\cdot&a&1&1&1&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\
a&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}_{\ddagger}
\begin{pmatrix}
\cdot&\cdot&1&1&1&1&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&1&\cdot&\cdot&1&1&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot&\cdot&1&1&1&1&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot&\cdot&1&1&1&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&1&1&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&1&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}_D
$$

$$
\begin{pmatrix}
\cdot&\cdot&1&1&1&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&1&1&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot&\cdot&1&1&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\
1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&1&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot&\cdot&a&1&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
a&1&\cdot&\cdot&1&1&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot&\cdot&1&1&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\
1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot&\cdot&1&1&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&1&\cdot&1&\cdot&\cdot\\
1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}_D
$$

$$
\begin{pmatrix}
\cdot&\cdot&1&1&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&1&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&1&\cdot&\cdot&1&\cdot&\cdot&1\\
1&\cdot&\cdot&\cdot&1&\cdot&\cdot&1\\
\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}_D
$$

‡ Continuous for $a = 1$, otherwise discrete.

Solutions with 18 zeros $\varepsilon^{18,1},\ldots,\varepsilon^{18,36}$

$$
\begin{pmatrix}
\cdot&\cdot&1&\cdot&\cdot&1&1&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&1&\cdot&1&\cdot&\cdot
\end{pmatrix}_C
\begin{pmatrix}
\cdot&\cdot&1&\cdot&\cdot&1&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot
\end{pmatrix}_C
\begin{pmatrix}
\cdot&\cdot&1&\cdot&\cdot&1&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&1&\cdot&1&\cdot&\cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot&\cdot&1&\cdot&\cdot&1&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&1&\cdot&1&\cdot&\cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&1&1&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot
\end{pmatrix}_C
$$

$$
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_C
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
$$

$$
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_C
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
$$

$$
\begin{pmatrix}
\cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
$$

$$
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot & \cdot & 1 & a & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\
a & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_*
$$

$$
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & 1 & \cdot & 1 & 1 \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\
\cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & a & \cdot & \cdot & 1 & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_*
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & a & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_*
$$

$$
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\
1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\
\cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot & \cdot & 1 & a & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot \\
a & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_*
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\
1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\
\cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_C
$$

$$
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
$$

* Continuous for $a = 1$, otherwise discrete.

Solutions with 19 zeros $\varepsilon^{19,1}, \ldots, \varepsilon^{19,45}$

$$
\begin{pmatrix}
\cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot
\end{pmatrix}_C
\begin{pmatrix}
\cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot
\end{pmatrix}_C
\begin{pmatrix}
\cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot
\end{pmatrix}_C
\begin{pmatrix}
\cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot
\end{pmatrix}_C
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_C
$$

$$
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_C
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_C
\begin{pmatrix}
\cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_D
\begin{pmatrix}
\cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}_C
$$

$$\begin{pmatrix} \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D$$

$$\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D$$

$$\begin{pmatrix} \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D$$

$$\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D$$

$$\begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D$$

$$\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D$$

$$\begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D$$

Solutions with 20 zeros $\varepsilon^{20,1},\ldots,\varepsilon^{20,42}$

$$\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C$$

$$\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D$$

$$\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C$$

$$\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\1&1&\cdot&\cdot&1&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \quad \begin{pmatrix}\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&\cdot&\cdot&1\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D$$

$$\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \quad \begin{pmatrix}\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&1&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D$$

$$\begin{pmatrix}\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \quad \begin{pmatrix}\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&1&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D$$

$$\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C$$

$$\begin{pmatrix}\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot&1&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C$$

$$\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D$$

Solutions with 21 zeros $\varepsilon^{21,1},\ldots,\varepsilon^{21,21}$

$$\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C$$

$$\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \quad \begin{pmatrix}\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D$$

$$\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C$$

$$\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \quad \begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C$$

$$\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C$$

Solutions with 22 zeros $\varepsilon^{22,1}, \ldots, \varepsilon^{22,7}$

$$\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C$$

$$\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C$$

Solutions with 23 zeros $\varepsilon^{23,1}$

$$\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C$$

Appendix B

The list of all contracted Lie algebras of the Pauli graded $sl(3,\mathbb{C})$ is presented in tabular form. In the table only non-decomposable non-Abelian parts of contracted Lie algebras are given. (For the structure of decomposable Lie algebras see (59)). Algebras are divided into classes according to dimensions of the derived series, the lower central series and the upper central series. For each listed Lie algebra we give its non-zero commutation relations, dimension of the algebra of derivations ($\mathcal{D}$) and the type T of contraction (C — continuous, D — discrete). For solvable non-nilpotent algebras the number τ of formal invariants and the dimension of the radical are added. For nilpotent algebras the Casimir operators are computed. Commutation relations are written in simplified form without the ω-coefficients, whenever this is possible. For all Lie algebras with dimensions lower than 6 and for 6-dimensional nilpotent algebras the names from [35] are also given.

Appendix B. Solvable Lie algebras

Series	Algebra	Commutation relations	τ	T	Nilradical	$\mathcal{D}$	Name
(430)(43)(0)	$\mathcal{L}'_{21,9}$	$[e_1,e_4]=e_2,\ [e_2,e_4]=e_3,\ [e_3,e_4]=e_1$	2	D	$3A_1$	6	$A_{4,6}(\frac{\pm 2}{\sqrt{3}},\frac{\mp 1}{\sqrt{3}})$
(530)(53)(0)	$\mathcal{L}'_{18,34}$	$[e_1,e_4]=e_2,\ [e_1,e_5]=e_3,\ [e_2,e_4]=e_3,\ [e_2,e_5]=e_1,\ [e_3,e_4]=e_1,\ [e_3,e_5]=e_2$	1	D	$3A_1$	6	$A_{5,33}(-1,-1)$
(640)(643)(12)	$\mathcal{L}'_{20,16}$	$[e_1,e_6]=e_3,\ [e_3,e_6]=e_4,\ [e_4,e_6]=e_1,\ [e_5,e_6]=e_2$	4	D	$5A_1$	10	
(740)(743)(12)	$\mathcal{L}'_{17,17}$	$[e_2,e_6]=e_3,\ [e_2,e_7]=e_4,\ [e_3,e_6]=e_4,\ [e_3,e_7]=e_2,$ $[e_4,e_6]=e_2,\ [e_4,e_7]=e_3,\ [e_5,e_6]=e_1$	3	D	$5A_1$	11	
(750)(7543)(123)	$\mathcal{L}'_{19,29}$	$[e_1,e_7]=e_3,\ [e_2,e_7]=e_5,\ [e_4,e_7]=e_2,\ [e_5,e_7]=e_4,\ [e_6,e_7]=e_1$	5	D	$6A_1$	12	
(760)(76)(0)	$\mathcal{L}'_{18,29}(a)$	$[e_1,e_7]=e_3,\ [e_2,e_7]=-ae_5,\ [e_3,e_7]=e_6,\ [e_4,e_7]=e_2,\ [e_5,e_7]=e_4,$ $[e_6,e_7]=e_1,\quad 0<\lvert a\rvert\le 1,\ a\neq\pm 1$	5	D	$6A_1$	12	
		$a=1$		C		12	
		$a=-1$		D		18	
(7630)(76)(0)	$\mathcal{L}'_{15,1}$	$[e_1,e_2]=-e_5,\ [e_1,e_3]=-e_6,\ [e_1,e_7]=e_3,\ [e_2,e_3]=e_4,\ [e_2,e_7]=e_1,$ $[e_3,e_7]=e_2,\ [e_4,e_7]=-e_6,\ [e_5,e_7]=e_4,\ [e_6,e_7]=e_5$	3	C	$\mathcal{L}'_{21,16}$	9	
(840)(843)(13)	$\mathcal{L}_{16,7}$	$[e_1,e_3]=-e_5,\ [e_1,e_4]=-e_8,\ [e_2,e_3]=e_7,\ [e_3,e_5]=e_8,$ $[e_3,e_8]=e_1,\ [e_4,e_5]=e_1,\ [e_4,e_6]=e_7,\ [e_4,e_8]=e_5$	4	D	$6A_1$	14	
(840)(843)(14)	$\mathcal{L}_{19,32}$	$[e_1,e_3]=e_5,\ [e_2,e_3]=e_7,\ [e_3,e_5]=e_8,\ [e_3,e_8]=e_1,\ [e_4,e_6]=e_7$	4	D	$\mathcal{L}'_{23,1}\oplus 4A_1$	17	
(850)(853)(23)	$\mathcal{L}_{16,5}$	$[e_1,e_3]=-e_5,\ [e_1,e_4]=-e_8,\ [e_2,e_3]=e_7,\ [e_2,e_4]=e_6,$ $[e_3,e_5]=e_8,\ [e_3,e_8]=e_1,\ [e_4,e_5]=e_1,\ [e_4,e_8]=e_5$	4	D	$6A_1$	13	
(850)(853)(24)	$\mathcal{L}_{19,27}$	$[e_1,e_3]=e_5,\ [e_2,e_3]=e_7,\ [e_2,e_4]=e_6,\ [e_3,e_5]=e_8,\ [e_3,e_8]=e_1$	4	D	$\mathcal{L}'_{23,1}\oplus 4A_1$	16	
(850)(8543)(123)	$\mathcal{L}_{15,6}(a)$	$[e_1,e_3]=e_5,\ [e_1,e_4]=-ae_8,\ [e_2,e_3]=-e_7,\ [e_2,e_4]=-e_6,\ [e_3,e_5]=e_8,$ $[e_3,e_6]=e_2,\ [e_3,e_7]=e_6,\ [e_4,e_6]=e_7,\ [e_4,e_7]=e_2$	4	D	$6A_1$	13	
(850)(8543)(124)	$\mathcal{L}_{18,28}$	$[e_1,e_3]=e_5,\ [e_2,e_3]=e_7,\ [e_2,e_4]=e_6,\ [e_3,e_5]=e_8,\ [e_3,e_8]=e_1,\ [e_4,e_6]=e_7$	4	D	$\mathcal{L}'_{22,1}\oplus 3A_1$	13	
(850)(8543)(134)	$\mathcal{L}_{18,27}$	$[e_1,e_3]=e_5,\ [e_2,e_3]=e_7,\ [e_2,e_4]=e_6,\ [e_3,e_5]=e_8,\ [e_3,e_7]=e_6,\ [e_3,e_8]=e_1$	4	D	$\mathcal{L}'_{23,1}\oplus 4A_1$	15	
(860)(86)(0)	$\mathcal{L}_{12,2}(a,b)$	$[e_1,e_3]=-e_5,\ [e_1,e_4]=-a(\omega+1)e_8,\ [e_2,e_3]=-(\omega+1)e_7,\ [e_2,e_4]=-e_6,$ $[e_3,e_5]=e_8,\ [e_3,e_6]=b(\omega+1)e_2,\ [e_3,e_7]=b(\omega+1)e_6,\ [e_3,e_8]=e_1,$ $[e_4,e_5]=a(\omega+1)e_1,\ [e_4,e_6]=e_7,\ [e_4,e_7]=be_2,\ [e_4,e_8]=a(\omega+1)e_5,$ $(\mathrm{Re}\ b>0)\vee(\mathrm{Re}\ b=0\wedge\mathrm{Im}\ b>0)$	4	D	$6A_1$	12	
		$a=b=I$		D		18	
		$a=b=1$		C		12	
(8630)(86)(0)	$\mathcal{L}_{9,1}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=(\omega+1)e_8,\ [e_1,e_5]=e_7,\ [e_1,e_6]=(\omega+1)e_4,$ $[e_1,e_7]=e_3,\ [e_1,e_8]=(\omega+1)e_6,\ [e_2,e_3]=(\omega+1)e_7,\ [e_2,e_4]=e_6,$ $[e_2,e_5]=(\omega+1)e_3,\ [e_2,e_6]=e_8,\ [e_2,e_7]=(\omega+1)e_5,\ [e_2,e_8]=e_4,$ $[e_4,e_6]=-e_7,\ [e_4,e_8]=-(\omega+1)e_5,\ [e_6,e_8]=-\omega e_3$	2	C	$\mathcal{L}'_{21,16}$	9	
(8710)(87)(1)	$\mathcal{L}_{15,4}$	$[e_1,e_3]=e_5,\ [e_1,e_6]=e_4,\ [e_2,e_6]=e_8,\ [e_2,e_7]=e_5,\ [e_3,e_6]=e_2,\ [e_4,e_6]=e_7,$ $[e_4,e_8]=e_5,\ [e_6,e_7]=-e_1,\ [e_6,e_8]=e_3$	2	C	$\mathcal{L}'_{21,20}$	11	
(8740)(87)(1)	$\mathcal{L}_{12,1}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=(\omega+1)e_8,\ [e_1,e_5]=e_7,\ [e_1,e_6]=(\omega+1)e_4,$ $[e_1,e_7]=e_3,\ [e_1,e_8]=(\omega+1)e_6,\ [e_3,e_6]=-(\omega+1)e_2,\ [e_4,e_6]=-e_7,$ $[e_4,e_7]=-e_2,\ [e_4,e_8]=-(\omega+1)e_5,\ [e_5,e_8]=\omega e_2,\ [e_6,e_8]=-\omega e_3$	2	C	(740)(7410)(147) $\mathcal{D}=15$	10	

Nilpotent Lie algebras

Series	Algebra	Commutation relations	Casimir operators	T	$\mathcal{D}$	Name
(310)(310)(13)	$\mathcal{L}'_{23,1}$	$[e_2,e_3]=e_1$	e_1	C	6	$A_{3,1}$
(420)(4210)(124)	$\mathcal{L}'_{22,1}$	$[e_1,e_4]=e_2,\ [e_3,e_4]=e_1$	$e_2,\ e_1^2-2e_2e_3$	C	7	$A_{4,1}$
(510)(510)(15)	$\mathcal{L}'_{22,7}$	$[e_2,e_4]=e_1,\ [e_3,e_5]=e_1$	e_1	C	15	$A_{5,4}$
(520)(520)(25)	$\mathcal{L}'_{22,3}$	$[e_3,e_4]=e_1,\ [e_3,e_5]=e_2$	$e_1,\ e_2,\ e_2e_4-e_1e_5$	C	13	$A_{5,1}$
(520)(5210)(135)	$\mathcal{L}'_{21,7}$	$[e_1,e_4]=e_2,\ [e_3,e_4]=e_1,\ [e_3,e_5]=e_2$	e_2	C	10	$A_{5,5}$
(530)(5320)(235)	$\mathcal{L}'_{21,2}$	$[e_1,e_4]=e_2,\ [e_1,e_5]=e_3,\ [e_4,e_5]=e_1$	$e_2,\ e_3,\ e_1^2+2e_2e_5-2e_3e_4$	C	10	$A_{5,3}$
(620)(620)(26)	$\mathcal{L}'_{21,19}$	$[e_3,e_4]=e_1,\ [e_3,e_6]=e_2,\ [e_4,e_5]=e_2$	$e_1,\ e_2$	C	17	$A_{6,4}$
(620)(6210)(146)	$\mathcal{L}'_{21,21}$	$[e_1,e_5]=e_2,\ [e_3,e_5]=e_1,\ [e_4,e_6]=e_2$	$e_2,\ e_1^2-2e_2e_3$	C	14	$A_{6,12}$
(630)(630)(36)	$\mathcal{L}'_{21,16}$	$[e_4,e_5]=e_1,\ [e_4,e_6]=e_2,\ [e_5,e_6]=e_3$	$e_1,\ e_2,\ e_3,\ e_1e_6-e_2e_5+e_3e_4$	C	18	$A_{6,3}$
(630)(6310)(136)	$\mathcal{L}'_{20,1}$	$[e_1,e_2]=e_4,\ [e_1,e_5]=e_3,\ [e_3,e_6]=e_4,\ [e_5,e_6]=e_2$	$e_4,\ e_4e_5-e_2e_3$	C	11	$A_{6,14}(-1)$
	$\mathcal{L}'_{20,2}$	$[e_1,e_5]=e_2,\ [e_3,e_4]=e_2,\ [e_4,e_5]=e_1,\ [e_4,e_6]=e_3$	$e_2,\ 2e_2e_6+e_3^2$	C	12	$A_{6,13}$
(630)(6310)(246)	$\mathcal{L}'_{21,1}$	$[e_1,e_5]=e_3,\ [e_4,e_5]=e_1,\ [e_4,e_6]=e_2$	$e_2,\ e_3$	C	13	$A_{6,7}$
	$\mathcal{L}'_{21,3}$	$[e_1,e_6]=e_3,\ [e_4,e_6]=e_1,\ [e_5,e_6]=e_2$	$e_2,\ e_3,\ e_1^2-2e_3e_4,\ e_3e_5-e_1e_2$	C	15	$A_{6,1}$
(630)(6320)(246)	$\mathcal{L}'_{20,8}$	$[e_1,e_4]=e_3,\ [e_1,e_5]=e_2,\ [e_4,e_5]=e_1,\ [e_4,e_6]=e_2$	$e_2,\ e_3$	C	13	$A_{6,9}$
(640)(64310)(1346)	$\mathcal{L}'_{19,2}$	$[e_2,e_6]=e_1,\ [e_3,e_5]=e_2,\ [e_3,e_6]=e_4,\ [e_4,e_5]=e_1,\ [e_5,e_6]=e_3$	$e_1,\ e_2e_4-e_1e_3$	C	10	$A_{6,18}(-1)$
(710)(710)(17)	$\mathcal{L}'_{21,20}$	$[e_2,e_4]=e_1,\ [e_3,e_6]=e_1,\ [e_5,e_7]=e_1$	e_1	C	28	
(720)(720)(27)	$\mathcal{L}'_{20,39}$	$[e_3,e_5]=e_2,\ [e_3,e_7]=e_1,\ [e_4,e_6]=e_1,\ [e_6,e_7]=e_2$	$e_1,\ e_2,\ e_1e_2e_7-e_1^2e_5+e_2^2e_4$	C	19	
	$\mathcal{L}'_{21,11}$	$[e_3,e_5]=e_1,\ [e_4,e_6]=e_2,\ [e_4,e_7]=e_1$	$e_1,\ e_2,\ e_2e_7-e_1e_6$	C	21	
(720)(7210)(157)	$\mathcal{L}'_{20,40}$	$[e_1,e_5]=e_2,\ [e_3,e_5]=e_1,\ [e_3,e_6]=e_2,\ [e_4,e_7]=e_2$	e_2	C	19	
(730)(730)(37)	$\mathcal{L}'_{21,12}$	$[e_4,e_6]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_7]=e_3$	$e_1,\ e_2,\ e_3$	C	20	
(730)(7310)(137)	$\mathcal{L}'_{19,44}$	$[e_1,e_6]=e_2,\ [e_3,e_5]=e_2,\ [e_4,e_6]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_7]=e_3$	e_2	D	13	
	$\mathcal{L}'_{20,42}$	$[e_1,e_6]=e_3,\ [e_2,e_5]=e_3,\ [e_4,e_6]=e_1,\ [e_5,e_7]=e_2$	$e_3,\ e_1^2-2e_3e_4,\ e_2^2+2e_3e_7$	D	14	
	$\mathcal{L}'_{19,1}$	$[e_1,e_2]=e_4,\ [e_1,e_5]=e_3,\ [e_1,e_6]=e_2,\ [e_3,e_7]=e_4,\ [e_5,e_7]=e_2$	$e_4,\ e_2^2-2e_4e_6,\ e_4e_5-e_2e_3$	C	15	
(730)(7310)(147)	$\mathcal{L}'_{19,7}$	$[e_1,e_6]=e_3,\ [e_2,e_4]=e_3,\ [e_4,e_6]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_7]=e_3$	e_3	C	16	
(730)(7310)(257)	$\mathcal{L}'_{20,19}$	$[e_1,e_6]=e_2,\ [e_4,e_6]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_7]=e_3$	$e_2,\ e_3,\ e_1^2e_3+2e_2^2e_5-2e_2e_3e_4$	D	15	
	$\mathcal{L}'_{20,5}$	$[e_1,e_6]=e_3,\ [e_4,e_6]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_7]=e_3$	$e_2,\ e_3,\ e_1^2+2e_2e_5-2e_3e_4$	C	17	
	$\mathcal{L}'_{20,41}$	$[e_1,e_6]=e_3,\ [e_4,e_6]=e_1,\ [e_5,e_6]=e_2,\ [e_5,e_7]=e_3$	$e_2,\ e_3,\ e_1^2-2e_3e_4$	C	18	
	$\mathcal{L}'_{20,14}$	$[e_1,e_6]=e_2,\ [e_4,e_6]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_6]=e_3$	$e_2,\ e_3,\ e_2e_5-e_1e_3$	C	19	
(730)(7320)(257)	$\mathcal{L}'_{19,19}$	$[e_1,e_4]=e_3,\ [e_1,e_5]=e_2,\ [e_4,e_5]=e_1,\ [e_4,e_6]=e_2,\ [e_6,e_7]=e_3$	$e_2,\ e_3,\ e_1^2e_3-2e_2^2e_7-2e_2e_3e_4+2e_3^2e_5$	D	14	
	$\mathcal{L}'_{19,17}$	$[e_1,e_4]=e_3,\ [e_1,e_6]=e_2,\ [e_4,e_6]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_6]=e_3$	$e_2,\ e_3,\ e_1e_2e_3-e_2^2e_5+e_3^2e_7$	C	16	
	$\mathcal{L}'_{20,9}$	$[e_1,e_4]=e_2,\ [e_1,e_6]=e_3,\ [e_4,e_6]=e_1,\ [e_5,e_7]=e_3$	$e_2,\ e_3,\ e_1^2+2e_2e_6-2e_3e_4$	D	16	
(740)(7410)(147)	$\mathcal{L}'_{18,13}$	$[e_1,e_6]=e_3,\ [e_2,e_5]=e_3,\ [e_4,e_7]=e_3,\ [e_5,e_6]=e_1,\ [e_5,e_7]=e_2,$ $[e_6,e_7]=e_4$	e_3	C	15	
(740)(7410)(247)	$\mathcal{L}'_{19,8}$	$[e_1,e_6]=e_3,\ [e_2,e_5]=e_3,\ [e_5,e_6]=e_1,\ [e_5,e_7]=e_2,\ [e_6,e_7]=e_4$	$e_3,\ e_4,\ 2e_1e_4+e_2^2+2e_3e_7$	C	16	
(740)(7410)(357)	$\mathcal{L}'_{20,6}$	$[e_1,e_6]=e_3,\ [e_5,e_6]=e_1,\ [e_5,e_7]=e_2,\ [e_6,e_7]=e_4$	$e_2,\ e_3,\ e_4$	C	18	
(740)(7420)(247)	$\mathcal{L}'_{18,9}$	$[e_1,e_5]=e_3,\ [e_1,e_6]=e_2,\ [e_4,e_5]=e_2,\ [e_4,e_7]=e_3,\ [e_5,e_6]=e_1,$ $[e_5,e_7]=e_4$	$e_2,\ e_3,\ e_1^2e_3+2e_2^2e_7-2e_2e_3e_5+e_2e_4^2+2e_3^2e_6$	D	11	
	$\mathcal{L}'_{19,9}$	$[e_1,e_6]=e_3,\ [e_2,e_5]=e_3,\ [e_2,e_7]=e_4,\ [e_5,e_6]=e_1,\ [e_5,e_7]=e_2$	$e_3,\ e_4,\ e_1^2e_4+e_2^2e_3+2e_3^2e_7-2e_3e_4e_5$	D	12	
	$\mathcal{L}'_{20,7}$	$[e_1,e_6]=e_3,\ [e_2,e_7]=e_4,\ [e_5,e_6]=e_1,\ [e_5,e_7]=e_2$	$e_3,\ e_4,\ e_1^2e_4+e_2^2e_3-2e_3e_4e_5$	D	13	
	$\mathcal{L}'_{19,4}$	$[e_1,e_2]=e_4,\ [e_1,e_5]=e_3,\ [e_3,e_5]=e_6,\ [e_3,e_7]=e_4,\ [e_5,e_7]=e_2$	$e_4,\ e_6,\ e_2e_3-e_4e_5+e_6e_7$	C	15	
	$\mathcal{L}'_{19,5}$	$[e_1,e_5]=e_2,\ [e_1,e_6]=e_4,\ [e_3,e_5]=e_4,\ [e_5,e_6]=e_1,\ [e_5,e_7]=e_3$	$e_2,\ e_4,\ e_3^2+2e_4e_7$	C	16	
	$\mathcal{L}'_{20,13}$	$[e_1,e_7]=e_3,\ [e_2,e_7]=e_4,\ [e_5,e_7]=e_1,\ [e_6,e_7]=e_2$	$e_3,\ e_4,\ e_1^2-2e_3e_5,\ e_2^2-2e_4e_6,\ e_2e_3-e_1e_4$	C	19	
(740)(7420)(357)	$\mathcal{L}'_{20,3}$	$[e_1,e_5]=e_2,\ [e_1,e_6]=e_4,\ [e_5,e_6]=e_1,\ [e_5,e_7]=e_3$	$e_2,\ e_3,\ e_4$	C	17	
(740)(74310)(1357)	$\mathcal{L}'_{18,1}$	$[e_1,e_7]=e_3,\ [e_2,e_5]=e_1,\ [e_2,e_7]=e_4,\ [e_4,e_5]=e_3,\ [e_5,e_6]=e_4,$ $[e_5,e_7]=e_2$	$e_3,\ e_2e_3-e_1e_4,\ e_4^2+2e_3e_6$	C	13	
(7510)(754210)(12457)	$\mathcal{L}'_{17,1}$	$[e_1,e_2]=-e_4,\ [e_1,e_7]=e_5,\ [e_2,e_6]=e_1,\ [e_2,e_7]=e_3,$ $[e_3,e_6]=e_5,\ [e_5,e_6]=e_4,\ [e_6,e_7]=e_2$	$e_4,\ e_5^2-2e_3e_4,\ e_2e_5-e_1e_3+e_4e_7$	C	12	

Series	Algebra	Commutation relations	Casimir operators	T	$\mathcal{D}$
(820)(820)(28)	$\mathcal{L}_{18,35}$	$[e_1,e_3]=e_5,\ [e_1,e_8]=e_6,\ [e_2,e_4]=e_6,$ $[e_2,e_7]=e_5,\ [e_3,e_7]=e_6,\ [e_4,e_8]=e_5$	$e_5,\ e_6$	C	22
	$\mathcal{L}_{19,42}$	$[e_1,e_3]=e_5,\ [e_1,e_8]=e_6,\ [e_2,e_4]=e_6,\ [e_2,e_7]=e_5,\ [e_3,e_7]=e_6$	$e_5,\ e_6$	C	24
	$\mathcal{L}_{20,32}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_2,e_6]=e_8,\ [e_2,e_7]=e_5$	$e_5,\ e_8,\ e_3e_8-e_4e_5,\ e_5e_6-e_7e_8$	C	26
	$\mathcal{L}_{20,38}$	$[e_1,e_3]=e_5,\ [e_1,e_8]=e_6,\ [e_2,e_4]=e_6,\ [e_3,e_7]=e_6$	$e_5,\ e_6$	C	28
(830)(830)(38)	$\mathcal{L}_{19,41}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_2,e_3]=e_7,\ [e_2,e_6]=e_8,\ [e_4,e_6]=e_7$	$e_5,\ e_7,\ e_8,\ e_1e_7^2-e_2e_5e_7-e_3e_8^2+e_4e_5e_8+e_6e_7e_8$	C	20
	$\mathcal{L}_{20,35}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_2,e_6]=e_8,\ [e_4,e_6]=e_7$	$e_5,\ e_7,\ e_8,\ e_2e_5e_7+e_3e_8^2-e_4e_5e_8$	C	23
	$\mathcal{L}_{20,31}$	$[e_1,e_3]=e_5,\ [e_1,e_6]=e_4,\ [e_2,e_6]=e_8,\ [e_2,e_7]=e_5$	$e_4,\ e_5,\ e_8,\ e_3e_4-e_5e_6+e_7e_8$	C	26
(830)(8310)(138)	$\mathcal{L}_{17,5}$	$[e_1,e_3]=e_5,\ [e_1,e_6]=e_4,\ [e_1,e_7]=e_3,\ [e_2,e_7]=e_5,$ $[e_2,e_8]=e_4,\ [e_4,e_8]=e_5,\ [e_6,e_8]=e_3$	$e_5,\ e_3e_4-e_5e_6$	D	16
	$\mathcal{L}_{18,3}$	$[e_1,e_3]=e_5,\ [e_1,e_6]=e_4,\ [e_1,e_7]=e_3,$ $[e_2,e_8]=e_4,\ [e_4,e_8]=e_5,\ [e_6,e_8]=e_3$	$e_5,\ e_3^2-2e_5e_7,\ e_3e_4-e_5e_6,\ e_4^2-2e_2e_5$	D	19
(830)(8310)(148)	$\mathcal{L}_{18,2}$	$[e_1,e_3]=e_5,\ [e_1,e_6]=e_4,\ [e_1,e_7]=e_3,$ $[e_2,e_7]=e_5,\ [e_4,e_8]=e_5,\ [e_6,e_8]=e_3$	$e_5,\ e_3e_4-e_5e_6$	C	18
(830)(8310)(158)	$\mathcal{L}_{19,3}$	$[e_1,e_3]=e_5,\ [e_1,e_6]=e_4,\ [e_2,e_7]=e_5,\ [e_4,e_8]=e_5,\ [e_6,e_8]=e_3$	$e_5,\ e_3e_4-e_5e_6$	C	20
(830)(8310)(268)	$\mathcal{L}_{18,24}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_2,e_3]=e_7,$ $[e_2,e_6]=e_8,\ [e_3,e_5]=e_8,\ [e_4,e_6]=e_7$	$e_7,\ e_8$	D	19
	$\mathcal{L}_{19,23}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_2,e_6]=e_8,\ [e_3,e_5]=e_8,\ [e_4,e_6]=e_7$	$e_7,\ e_8$	D	20
	$\mathcal{L}_{19,24}$	$[e_1,e_3]=e_5,\ [e_2,e_3]=e_7,\ [e_2,e_6]=e_8,\ [e_3,e_5]=e_8,\ [e_4,e_6]=e_7$	$e_7,\ e_8,\ 2e_1e_8+e_5^2,\ e_2e_7e_8-e_4e_8^2+e_5e_7^2$	D	20
	$\mathcal{L}_{19,31}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_2,e_3]=e_7,\ [e_3,e_5]=e_8,\ [e_4,e_6]=e_7$	$e_7,\ e_8,\ e_2e_8+e_5e_7,\ 2e_1e_7e_8+e_5^2e_7+2e_6e_8^2$	D	21
	$\mathcal{L}_{20,11}$	$[e_1,e_3]=e_5,\ [e_2,e_6]=e_8,\ [e_3,e_5]=e_8,\ [e_4,e_6]=e_7$	$e_7,\ e_8,\ e_2e_7-e_4e_8,\ 2e_1e_8+e_5^2$	D	21
	$\mathcal{L}_{20,17}$	$[e_1,e_3]=e_5,\ [e_2,e_3]=e_7,\ [e_3,e_5]=e_8,\ [e_4,e_6]=e_7$	$e_7,\ e_8,\ e_2e_8+e_5e_7,\ 2e_1e_8+e_5^2$	D	23
	$\mathcal{L}_{19,43}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_2,e_3]=e_7,\ [e_2,e_6]=e_8,\ [e_3,e_5]=e_8$	$e_7,\ e_8$	C	24
(830)(8320)(268)	$\mathcal{L}_{17,14}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,\ [e_2,e_3]=e_7,$ $[e_2,e_6]=e_8,\ [e_3,e_5]=e_8,\ [e_4,e_6]=e_7$	$e_7,\ e_8$	D	16
	$\mathcal{L}_{18,20}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,$ $[e_2,e_6]=e_8,\ [e_3,e_5]=e_8,\ [e_4,e_6]=e_7$	$e_7,\ e_8$	D	18
	$\mathcal{L}_{18,18}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,$ $[e_2,e_3]=e_7,\ [e_2,e_6]=e_8,\ [e_3,e_5]=e_8$	$e_7,\ e_8$	D	20
	$\mathcal{L}_{19,21}$	$[e_1,e_3]=e_5,\ [e_1,e_5]=e_7,\ [e_2,e_6]=e_8,\ [e_3,e_5]=e_8,\ [e_4,e_6]=e_7$	$e_7,\ e_8,\ e_2e_7-e_4e_8,\ 2e_1e_8-2e_3e_7+e_5^2$	D	20
	$\mathcal{L}_{19,18}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,\ [e_2,e_6]=e_8,\ [e_3,e_5]=e_8$	$e_7,\ e_8$	D	22
(840)(840)(48)	$\mathcal{L}_{20,33}$	$[e_1,e_3]=e_5,\ [e_1,e_6]=e_4,\ [e_2,e_3]=e_7,\ [e_2,e_6]=e_8$	$e_4,\ e_5,\ e_7,\ e_8$	C	24
(840)(8410)(248)	$\mathcal{L}_{18,36}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_2,e_3]=e_7,$ $[e_2,e_4]=e_6,\ [e_2,e_6]=e_8,\ [e_3,e_5]=e_8$	$e_7,\ e_8$	D	18
	$\mathcal{L}_{19,45}$	$[e_1,e_3]=e_5,\ [e_2,e_3]=e_7,\ [e_2,e_4]=e_6,\ [e_2,e_6]=e_8,\ [e_3,e_5]=e_8$	$e_7,\ e_8,\ e_5^2+2e_1e_8,\ e_6^2-2e_4e_8$	D	19
(840)(8410)(258)	$\mathcal{L}_{18,10}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_6]=e_4,$ $[e_2,e_3]=e_7,\ [e_2,e_6]=e_8,\ [e_3,e_5]=e_8$	$e_7,\ e_8$	D	19
	$\mathcal{L}_{19,6}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_6]=e_4,\ [e_2,e_3]=e_7,\ [e_3,e_5]=e_8$	$e_7,\ e_8,\ e_4^2-2e_6e_8,\ e_2e_8+e_5e_7$	D	20
(840)(8410)(368)	$\mathcal{L}_{19,25}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_2,e_3]=e_7,\ [e_2,e_4]=e_6,\ [e_3,e_5]=e_8$	$e_6,\ e_7,\ e_8,\ 2e_1e_6e_8-2e_2e_8^2+e_5^2e_6-2e_5e_7e_8$	D	20
	$\mathcal{L}_{19,13}$	$[e_1,e_3]=e_5,\ [e_1,e_6]=e_4,\ [e_2,e_3]=e_7,\ [e_2,e_6]=e_8,\ [e_3,e_5]=e_8$	$e_4,\ e_7,\ e_8,\ 2e_1e_8^2-2e_2e_4e_8-2e_4e_5e_7+e_5^2e_8$	C	21
	$\mathcal{L}_{20,12}$	$[e_1,e_3]=e_5,\ [e_2,e_3]=e_7,\ [e_2,e_4]=e_6,\ [e_3,e_5]=e_8$	$e_6,\ e_7,\ e_8,\ 2e_1e_8+e_5^2$	D	22
	$\mathcal{L}_{20,4}$	$[e_1,e_3]=e_5,\ [e_1,e_6]=e_4,\ [e_2,e_3]=e_7,\ [e_3,e_5]=e_8$	$e_4,\ e_7,\ e_8,\ e_2e_8+e_5e_7$	C	23
(840)(8420)(248)	$\mathcal{L}_{16,4}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,\ [e_2,e_3]=e_7,$ $[e_2,e_4]=e_6,\ [e_2,e_6]=e_8,\ [e_3,e_5]=e_8,\ [e_4,e_6]=e_7$	$e_7,\ e_8$	D	12
	$\mathcal{L}_{17,16}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,\ [e_2,e_4]=e_6,$ $[e_2,e_6]=e_8,\ [e_3,e_5]=e_8,\ [e_4,e_6]=e_7$	$e_7,\ e_8$	D	14

Series	Algebra	Commutation relations	Casimir operators	T	$\mathcal{D}$
	$\mathcal{L}_{17,15}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,\ [e_2,e_3]=e_7,$ $[e_2,e_4]=e_6,\ [e_2,e_6]=e_8,\ [e_3,e_5]=e_8$	$e_7,\ e_8$	D	15
	$\mathcal{L}_{18,21}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,\ [e_2,e_4]=e_6,$ $[e_2,e_6]=e_8,\ [e_3,e_5]=e_8$	$e_7,\ e_8$	D	16
	$\mathcal{L}_{18,22}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,\ [e_2,e_4]=e_6,$ $[e_3,e_5]=e_8,\ [e_4,e_6]=e_7$	$e_7,\ e_8,\ 2e_2e_7+e_6^2,\ 2e_1e_7e_8-2e_3e_7^2+e_5^2e_7+2e_6e_8^2$	D	16
	$\mathcal{L}_{18,23}$	$[e_1,e_3]=e_5,\ [e_1,e_5]=e_7,\ [e_2,e_4]=e_6,\ [e_2,e_6]=e_8,$ $[e_3,e_5]=e_8,\ [e_4,e_6]=e_7$	$e_7,\ e_8,\ 2e_1e_8-2e_3e_7+e_5^2,\ 2e_2e_7-2e_4e_8+e_6^2$	D	16
	$\mathcal{L}_{18,26}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_2,e_3]=e_7,\ [e_2,e_4]=e_6,$ $[e_3,e_5]=e_8,\ [e_4,e_6]=e_7$	$e_7,\ e_8,\ 2e_1e_7e_8+e_5^2e_7+2e_6e_8^2,\ 2e_2e_7e_8+2e_5e_7^2+e_6^2e_8$	D	16
	$\mathcal{L}_{17,2}$	$[e_1,e_3]=e_5,\ [e_1,e_6]=e_4,\ [e_1,e_7]=e_3,\ [e_3,e_6]=e_2,$ $[e_4,e_7]=e_2,\ [e_4,e_8]=e_5,\ [e_6,e_8]=e_3$	$e_2,\ e_5,\ e_1e_2-e_3e_4+e_5e_6,\ 2e_2e_8+e_3^2-2e_5e_7$	C	17
	$\mathcal{L}_{19,22}$	$[e_1,e_3]=e_5,\ [e_1,e_5]=e_7,\ [e_2,e_4]=e_6,\ [e_2,e_6]=e_8,\ [e_3,e_5]=e_8$	$e_7,\ e_8,\ 2e_4e_8-e_6^2,\ 2e_1e_8-2e_3e_7+e_5^2$	D	17
	$\mathcal{L}_{19,28}$	$[e_1,e_3]=e_5,\ [e_2,e_3]=e_7,\ [e_2,e_4]=e_6,\ [e_3,e_5]=e_8,\ [e_4,e_6]=e_7$	$e_7,\ e_8,\ 2e_1e_8+e_5^2,\ 2e_2e_7e_8+2e_5e_7^2+e_6^2e_8$	D	17
(840)(8420)(258)	$\mathcal{L}_{16,3}(a)$	$[e_1,e_3]=-ae_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,\ [e_1,e_6]=e_4,$ $[e_2,e_3]=e_7,\ [e_2,e_6]=e_8,\ [e_3,e_5]=e_8,\ [e_4,e_6]=e_7,$ $0<\lvert a\rvert\leq 1$	$e_7,\ e_8$	D	16
	$\mathcal{L}_{17,9}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,\ [e_1,e_6]=e_4,$ $[e_2,e_3]=e_7,\ [e_3,e_5]=e_8,\ [e_4,e_6]=e_7$	$e_7,\ e_8$	D	16
	$\mathcal{L}_{17,12}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_6]=e_4,\ [e_2,e_3]=e_7,$ $[e_2,e_6]=e_8,\ [e_3,e_5]=e_8,\ [e_4,e_6]=e_7$	$e_7,\ e_8$	D	16
	$\mathcal{L}_{18,12}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_6]=e_4,\ [e_2,e_3]=e_7,$ $[e_3,e_5]=e_8,\ [e_4,e_6]=e_7$	$e_7,\ e_8,\ e_2e_8+e_5e_7,\ 2e_1e_7e_8-e_4^2e_8+e_5^2e_7+2e_6e_8^2$	D	17
	$\mathcal{L}_{18,14}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_6]=e_4,\ [e_2,e_6]=e_8,$ $[e_3,e_5]=e_8,\ [e_4,e_6]=e_7$	$e_7,\ e_8$	D	17
	$\mathcal{L}_{18,17}$	$[e_1,e_3]=e_5,\ [e_1,e_6]=e_4,\ [e_2,e_3]=e_7,\ [e_2,e_6]=e_8,$ $[e_3,e_5]=e_8,\ [e_4,e_6]=e_7$	$e_7,\ e_8,\ 2e_1e_7e_8-e_4^2e_8+e_5^2e_7,\ e_2e_7e_8-e_4e_8^2+e_5e_7^2$	D	17
	$\mathcal{L}_{19,15}$	$[e_1,e_3]=e_5,\ [e_1,e_6]=e_4,\ [e_2,e_3]=e_7,\ [e_3,e_5]=e_8,\ [e_4,e_6]=e_7$	$e_7,\ e_8,\ e_2e_8+e_5e_7,\ 2e_1e_7e_8-e_4^2e_8+e_5^2e_7$	D	18
	$\mathcal{L}_{17,7}(a)$	$[e_1,e_3]=-ae_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,\ [e_1,e_6]=e_4,$ $[e_2,e_3]=e_7,\ [e_2,e_6]=e_8,\ [e_3,e_5]=e_8,\quad 0\neq a\neq 1$	$e_7,\ e_8$	D	19
		$a=1$	-//-,-//-	C	19
	$\mathcal{L}_{18,6}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,\ [e_1,e_6]=e_4,$ $[e_2,e_3]=e_7,\ [e_3,e_5]=e_8$	$e_7,\ e_8,\ e_4^2-2e_6e_8,\ e_2e_8^2-e_4e_7^2+e_5e_7e_8$	D	19
	$\mathcal{L}_{18,7}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,\ [e_1,e_6]=e_4,$ $[e_2,e_6]=e_8,\ [e_3,e_5]=e_8$	$e_7,\ e_8$	D	20
	$\mathcal{L}_{18,25}(a)$	$[e_1,e_3]=e_5,\ [e_1,e_4]=-ae_8,\ [e_2,e_3]=e_7,\ [e_2,e_4]=e_6,$ $[e_3,e_5]=e_8,\ [e_3,e_7]=e_6,\quad 0<\lvert a\rvert\leq 1,\ a\neq\pm 1$	$e_6,\ e_8,\ e_5e_6-e_7e_8,$ $2e_1e_6e_8+2ae_2e_8^2+(1-a)e_5^2e_6+2ae_5e_7e_8$	D	20
		$a=-1$	-//-,-//-,-//-,-//-	D	22
		$a=1$	-//-,-//-,-//-, $e_1e_6+e_2e_8+e_5e_7$	C	20
		$a=0$	-//-,-//-,-//-, $2e_1e_8+e_5^2$	D	21
(840)(8420)(368)	$\mathcal{L}_{18,19}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,\ [e_2,e_3]=e_7,$ $[e_2,e_4]=e_6,\ [e_3,e_5]=e_8$	$e_6,\ e_7,\ e_8,$ $2e_1e_6e_8-2e_2e_8^2-2e_3e_6e_7+2e_4e_7^2+e_5^2e_6-2e_5e_7e_8$	D	16
	$\mathcal{L}_{18,16}$	$[e_1,e_3]=e_5,\ [e_1,e_5]=e_7,\ [e_1,e_6]=e_4,\ [e_2,e_3]=e_7,$ $[e_2,e_6]=e_8,\ [e_3,e_5]=e_8$	$e_4,\ e_7,\ e_8,$ $2e_1e_8^2-2e_2e_4e_8-2e_3e_7e_8-2e_4e_5e_7+e_5^2e_8+2e_6e_7^2$	D	19

Series	Algebra	Commutation relations	Casimir operators	T	$\mathcal{D}$
	$\mathcal{L}_{19,20}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,\ [e_2,e_4]=e_6,\ [e_3,e_5]=e_8$	$e_6,\ e_7,\ e_8,\ 2e_1e_6e_8-2e_2e_8^2-2e_3e_6e_7+e_5^2e_6$	D	19
	$\mathcal{L}_{19,10}$	$[e_1,e_3]=e_5,\ [e_1,e_5]=e_7,\ [e_1,e_6]=e_4,\ [e_2,e_3]=e_7,\ [e_3,e_5]=e_8$	$e_4,\ e_7,\ e_8,\ e_2e_4e_8+e_4e_5e_7-e_6e_7^2$	C	21
	$\mathcal{L}_{19,11}$	$[e_1,e_3]=e_5,\ [e_1,e_5]=e_7,\ [e_1,e_6]=e_4,\ [e_2,e_6]=e_8,\ [e_3,e_5]=e_8$	$e_4,\ e_7,\ e_8,\ 2e_1e_8-2e_2e_4-2e_3e_7+e_5^2$	D	21
(840)(84310)(1368)	$\mathcal{L}_{16,1}(a)$	$[e_1,e_3]=e_5,\ [e_1,e_6]=e_4,\ [e_1,e_7]=e_3,\ [e_1,e_8]=e_6,\ [e_2,e_7]=e_5$ $[e_2,e_8]=-ae_4,\ [e_4,e_8]=e_5,\ [e_6,e_8]=e_3,\ 0\neq a\neq 1$	$e_5,\ e_3e_4-e_5e_6$	D	16
		$a=1$	-//-,-//-	C	16
	$\mathcal{L}_{17,4}$	$[e_1,e_3]=e_5,\ [e_1,e_6]=e_4,\ [e_1,e_7]=e_3,\ [e_1,e_8]=e_6,$ $[e_2,e_8]=e_4,\ [e_4,e_8]=e_5,\ [e_6,e_8]=e_3$	$e_5,\ e_3^2-2e_5e_7,\ e_3e_4-e_5e_6,\ 2e_2e_5-e_4^2$	D	16
(840)(84310)(1468)	$\mathcal{L}_{17,3}$	$[e_1,e_3]=e_5,\ [e_1,e_6]=e_4,\ [e_1,e_7]=e_3,\ [e_1,e_8]=e_6,$ $[e_2,e_7]=e_5,\ [e_4,e_8]=e_5,\ [e_6,e_8]=e_3$	$e_5,\ e_3e_4-e_5e_6$	D	17
(840)(84310)(1568)	$\mathcal{L}_{18,4}$	$[e_1,e_3]=e_5,\ [e_1,e_6]=e_4,\ [e_1,e_8]=e_6,\ [e_2,e_7]=e_5,$ $[e_4,e_8]=e_5,\ [e_6,e_8]=e_3$	$e_5,\ e_3e_4-e_5e_6$	D	19
(850)(8520)(258)	$\mathcal{L}_{15,5}(a,b)$	$[e_1,e_3]=-ae_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,\ [e_1,e_6]=e_4,$ $[e_2,e_3]=e_7,\ [e_2,e_6]=e_8,\ [e_3,e_5]=-be_8,\ [e_3,e_6]=e_2,$ $[e_4,e_6]=e_7,\quad a\neq 0$	$e_7,\ e_8$	D	16
		$a=-1,\ b=0$	-//-,-//-	D	17
		$a=1,\ b=1$	-//-,-//-	C	18
	$\mathcal{L}_{17,8}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,\ [e_1,e_6]=e_4,$ $[e_2,e_3]=e_7,\ [e_3,e_5]=e_8,\ [e_3,e_6]=e_2$	$e_7,\ e_8,\ e_2e_8^2-e_4e_7^2+e_5e_7e_8,\ e_2^2e_8-e_4^2e_7+2e_6e_7e_8$	D	16
	$\mathcal{L}_{17,10}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,\ [e_1,e_6]=e_4,$ $[e_2,e_6]=e_8,\ [e_3,e_5]=e_8,\ [e_3,e_6]=e_2$	$e_7,\ e_8$	D	16
	$\mathcal{L}_{17,13}(a)$	$[e_1,e_3]=ae_5,\ [e_1,e_4]=e_8,\ [e_1,e_6]=e_4,\ [e_2,e_3]=e_7,$ $[e_3,e_5]=e_8,\ [e_3,e_6]=e_2,\ [e_4,e_6]=e_7,\ 0<\lvert a\rvert\leq 1,\ a\neq\pm 1$	$e_7,\ e_8,\ e_2e_8+e_5e_7,$ $2e_1e_7^2+(1-a)e_2^2e_8+-2ae_2e_5e_7+e_4^2e_7-2e_6e_7e_8$	D	17
		$a=-1$	-//-,-//-,-//-,-//-	D	19
		$a=1$	-//-,-//-,-//-, $2e_1e_7-2e_2e_5-e_4^2+2e_6e_8$	D	17
	$\mathcal{L}_{18,11}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_6]=e_4,\ [e_2,e_3]=e_7,$ $[e_3,e_5]=e_8,\ [e_3,e_6]=e_2$	$e_7,\ e_8,\ e_2e_8+e_5e_7,\ e_2^2e_8-e_4^2e_7+2e_6e_7e_8$	D	17
(850)(8520)(358)	$\mathcal{L}_{17,11}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,\ [e_1,e_6]=e_4,$ $[e_3,e_5]=e_8,\ [e_3,e_6]=e_2,\ [e_4,e_6]=e_7$	$e_2,\ e_7,\ e_8,$ $2e_1e_7e_8+2e_2e_4e_7-2e_2e_5e_8-2e_3e_7^2-e_4^2e_8+e_5^2e_7+2e_6e_8^2$	D	16
	$\mathcal{L}_{18,15}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_6]=e_4,\ [e_3,e_5]=e_8,$ $[e_3,e_6]=e_2,\ [e_4,e_6]=e_7$	$e_2,\ e_7,\ e_8,\ 2e_1e_7e_8-2e_2e_5e_8-e_4^2e_8+e_5^2e_7+2e_6e_8^2$	D	17
	$\mathcal{L}_{19,16}$	$[e_1,e_3]=e_5,\ [e_1,e_6]=e_4,\ [e_3,e_5]=e_8,\ [e_3,e_6]=e_2,\ [e_4,e_6]=e_7$	$e_2,\ e_7,\ e_8,\ 2e_1e_7e_8-e_4^2e_8+e_5^2e_7$	D	18
	$\mathcal{L}_{18,8}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_5]=e_7,\ [e_1,e_6]=e_4,$ $[e_3,e_5]=e_8,\ [e_3,e_6]=e_2$	$e_2,\ e_7,\ e_8,\ 2e_2e_4e_7-2e_2e_5e_8-e_4^2e_8+2e_6e_8^2$	C	20
	$\mathcal{L}_{19,14}$	$[e_1,e_3]=e_5,\ [e_1,e_6]=e_4,\ [e_2,e_3]=e_7,\ [e_3,e_5]=e_8,\ [e_3,e_6]=e_2$	$e_4,\ e_7,\ e_8,\ e_2e_8+e_5e_7$	C	22
(850)(8520)(468)	$\mathcal{L}_{19,12}$	$[e_1,e_3]=e_5,\ [e_1,e_5]=e_7,\ [e_1,e_6]=e_4,\ [e_3,e_5]=e_8,\ [e_3,e_6]=e_2$	$e_2,\ e_4,\ e_7,\ e_8$	C	22
(850)(8530)(358)	$\mathcal{L}_{18,5}$	$[e_1,e_3]=e_5,\ [e_1,e_6]=e_4,\ [e_3,e_6]=e_2,\ [e_4,e_6]=e_7,$ $[e_4,e_8]=e_5,\ [e_6,e_8]=e_3$	$e_2,\ e_5,\ e_7,\ e_1e_2-e_3e_4+e_5e_6-e_7e_8$	C	19
(850)(85310)(1358)	$\mathcal{L}_{17,6}$	$[e_1,e_3]=e_5,\ [e_1,e_6]=e_4,\ [e_2,e_6]=e_8,\ [e_2,e_7]=e_5,$ $[e_4,e_6]=e_7,\ [e_4,e_8]=e_5,\ [e_6,e_8]=e_3$	$e_5,\ e_3e_4-e_5e_6+e_7e_8$	C	16
(8510)(854210)(12468)	$\mathcal{L}_{15,2}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_6]=e_4,\ [e_1,e_7]=-e_3,$ $[e_2,e_6]=e_8,\ [e_2,e_7]=e_5,\ [e_4,e_6]=e_7,\ [e_4,e_8]=e_5,\ [e_6,e_8]=e_3$	$e_5,\ e_3e_4-e_5e_6+e_7e_8$	C	13
(8620)(865320)(23568)	$\mathcal{L}_{15,3}$	$[e_1,e_3]=e_5,\ [e_1,e_4]=e_8,\ [e_1,e_6]=e_4,\ [e_1,e_7]=e_3,\ [e_3,e_6]=e_2,$ $[e_4,e_6]=-e_7,\ [e_4,e_7]=e_2,\ [e_4,e_8]=e_5,\ [e_6,e_8]=e_3$	$e_2,\ e_5,\ 2e_2e_8+e_3^2-2e_5e_7,\ e_1e_2-e_3e_4+e_5e_6+e_7e_8$	C	14

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